HIGH-GAIN SAMPLED-DATA CONTROL OF INTERCONNECTED SYSTEMS

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND

ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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January 2002

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

HIGH-GAIN SAMPLED-DATA CONTROL OF INTERCONNECTED SYSTEMS

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Stabilization of interconnected systems using adaptive, decentralized, high-gain, sampled-data controllers is considered. Main applications of high-gain methodology to various systems under modeling uncertainties are reviewed. Then, sampled-data, high-gain and decentralized control techniques are combined to find a solution to stabilization of interconnected systems, while satisfying the overall synchronization of the whole system. It is shown that overall system can be stabilized in continuous and discrete time domains by applying an adaptation mechanism for perturbations with unknown bounds.

Keywords: interconnected system, subsystem, high-gain control, decentralized control, sampled-data control, perturbation, adaptation, state feedback, output feedback

ÖZET

BİLEŞİK SİSTEMLERİN YÜKSEK KAZANÇLI ÖRNEKLENMİŞ KONTROLU

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Ocak 2001

Bileşik sistemlerin uyumlu, ayrışık, yüksek kazançlı, örneklenmiş veri geribeslemesi ile kararlılaştırılması incelenmiştir. Yüksek kazanç yönteminin modelleme belirsizliği olan çeşitli sistemlerdeki ana uygulamaları gözden geçirilmiştir. Daha sonra örneklenmiş veri, yüsek kazanç ve ayrışık geribesleme teknikleri bileşik sistemlerin karalaştırılmaısı için birlikte kullanılmıştır, aynı zamanda toplam sistemin eşgüdümü sağlanmıştır. Toplam sistemin sınırları bilinmeyen belirsizliklere karşı sürekli ve örneklenmiş zaman boyutlarında kararlılaştırılabildiği uyum mekanimasının uygulanması ile gösterilmiştir.

Anahtar kelimeler: bileşik sistem, alt sistem, yüksek kazançlı kontrol, ayrışık kontrol, örneklenmiş veri kontrolu, belirsizlik, uyumluluk, durum geribeslemesi, çıktı geribeslemesi

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Chapter 1

INTRODUCTION

High-gain control is a powerful tool to stabilize complex systems under additive perturbations and/or with modeling uncertainties that can be represented as additive perturbations. The basic idea behind high-gain control is to achieve a sufficiently high degree of stability of a nominal system to overcome any destabilizing effect of perturbations.

High-gain control has its roots in root-locus method and the small gain theorem [19]. As an illustration of the application of high-gain control, consider a single-input/single-output (SISO) system described as

$$\begin{aligned} \dot{x} &= Ax + bu + bg^T x \\ y &= c^T x \end{aligned}$$

where the term $bg^T x$ represents linear additive perturbations that satisfy the so called matching conditions [6]. Let

$$h(s) = c^T (sI - A)^{-1} b = \frac{q(s)}{d(s)}$$

and

$$g(s) = g^T (sI - A)^{-1} b = \frac{p(s)}{d(s)}$$

The transfer function of the perturbed system is calculated as

$$\begin{aligned}
h_p(s) &= c^T(sI - A - bg^T)^{-1}b \\
&= c^T(sI - A)^{-1}[I - bg^T(sI - A)^{-1}]^{-1}b \\
&= c^T(sI - A)^{-1}b[1 - g^T(sI - A)^{-1}b]^{-1} \\
&= \frac{h(s)}{1 - g(s)} \\
&= \frac{q(s)}{d(s) - p(s)}
\end{aligned}$$

Comparing $h(s)$ and $h_p(s)$, we observe that matching perturbations affect only the poles of the system but not the zeros. It is precisely this nature of the perturbations that allow for achieving stability by means of high-gain feedback control. For illustration purposes, let us assume that

$$h(s) = \frac{q(s)}{d(s)} = \frac{s^{n-1} + \cdots + q_{n-1}}{s^n + d_1 s^{n-1} + \cdots + d_n},$$

that is, $h(s)$ has relative degree one, and that $q(s)$ is a stable polynomial. Then, under constant output feedback

$$u = -kx$$

the closed-loop transfer function of the perturbed system becomes

$$\hat{h}_p(s) = \frac{h_p(s)}{1 + kh_p(s)} = \frac{q(s)}{d(s) - p(s) + kq(s)}$$

so that closed-loop characteristic polynomial is

$$\hat{d}_p(s) = d(s) - p(s) + kq(s)$$

Since

$$deg(d - p) = deg(q) + 1$$

it follows that as $k \to \infty$, $n - 1$ zeros of $\hat{d}_p(s)$ approach the stable zeros of $q(s)$ and the nth one tends to $-\infty$. In other words, there exists a critical gain k_c such that $\hat{d}_p(s)$ is stable for all $k > k_c$. The value of k_c depends on the location of zeros of $q(s)$ as well those of $d(s) - p(s)$, which in turn, depend on the perturbations.

An alternative interpretation of the above result can be provided in the light of the small-gain theorem. Expressing $\hat{h}_p(s)$ as

$$\hat{h}_p(s) = \frac{\frac{q(s)}{d(s)+kq(s)}}{1 - \frac{q(s)}{d(s)+kq(s)}\frac{p(s)}{q(s)}} = \frac{\hat{h}(s)}{1 - \hat{h}(s)r(s)}$$

we observe that the closed-loop perturbed system can be viewed as a feedback connection of two systems with transfer functions

$$\hat{h}(s) = \frac{q(s)}{d(s) + kq(s)} = \frac{h(s)}{1 + kh(s)}$$

and

$$r(s) = \frac{p(s)}{q(s)}$$

respectively. Since $q(s)$ is stable by assumption, $r(s)$ represents a stable system. On the other hand, by choosing k sufficiently large, not only $\hat{h}(s)$ can be made stable, but also $\| \hat{h}(s) \|_\infty$ can be made arbitrarily small. Then, the small-gain theorem guarantees stability of the closed-loop perturbed system for sufficiently large k.

Both the root-locus and the small-gain interpretations of high-gain feedback remain valid even when the relative degree of $h(s)$ is larger than one, which necessitates the use of dynamic output feedback. A further point worth to be mentioned is that since the roles of the input and output are symmetric as far as output feedback is concerned, the argument above can be repeated for perturbations of the form $hc^T x$, that is, perturbations satisfying the matching conditions on the output side. Both types of perturbations fall in a class termed "structured perturbations" [18].

The idea introduced above is applicable to single-input/single-output (SISO) systems whose zeros are stable and whose relative degree, high frequency gain and perturbation bounds are known. For multi-input/multi-output (MIMO) systems same requirements are valid. In [3], the idea was improved one step further, and systems with relative degree one were stabilized without knowing the bounds of perturbation by adaptively adjusting the controller gain. In [10], systems with higher relative degree were considered, where the gain parameter was increased adaptively at discrete instants.

High-gain technique is also used with sampled-data controllers by keeping the same assumptions on nominal system and perturbations as in the continuous-time case. In [8], SISO systems with controllers that operate on the sampled values of output have been stabilized. However, sampling action changes the perturbation structure such that perturbations are exponentiated in converting to discrete-time. To solve this problem sampling period was chosen as reciprocal of the gain, so that perturbations simply do not have enough time between successive sampling instants to cause instability.

Interconnected systems have been worked on by considering interconnections between subsystems as perturbation sources. The difficulty here is to achieve overall stability by using decentralized controllers. It is well-established [15] that once the interconnections satisfy matching conditions, then decentralization of the control does not create additional difficulty in stabilization by state-feedback. In [8], this nature of decentralized control was exploited to stabilize interconnected systems using sampled-data high-gain state feedback.

Applying high-gain sampled-data output-feedback control to interconnected systems is the main topic of the thesis. As in the continuous-time case, each subsystem is considered as a separate system with its own inner dynamics and sampled-data dynamic output feedback controllers are designed according to these inner structure. Parallel to single system controller, gains are chosen as the reciprocal of sampling period. Thus, sampling periods of subsystems are not necessarily the same and to be able talk on an overall stability of the whole system, synchronization is necessary. Then, question arises as: How can synchronization be satisfied without disturbing the gain constraints of the system? To answer this problem, all the sampling intervals of subsystems are chosen to be synchronized on a common sampling period, that is, common sampling period is an integer multiple of each subsystem periods, by keeping in mind the reciprocal relation between sampling period and gain. On the other hand, common sampling interval is not static, that means, it changes with time for adaptive adjustment.

Similar to the previous cases, in sampled-data decentralized control, an adaptation mechanism is employed against unknown interconnection bounds. However, applying the same adaptation rule as in the previous cases, can cause uncontrolled increase in gain parameter. This can prevent us from satisfying overall continuous-time stability. Hence, gain parameter is kept constant for a fixed time interval, which provides us overall continuous time and discrete time stabilities together.

The organization of the thesis is as follows:

In Chapter 2, the important high-gain applications are reviewed. The basic canonical forms that are used throughout the high-gain analyses are explained before single input state feedback case. Then a perturbed SISO system is stabilized with high-gain dynamic output feedback. Unbounded perturbations are beaten by applying an adaptation mechanism to increase the gain in a required way. Afterwards, interconnected systems are stabilized in continuous-time.

Chapter 3 is devoted to the analysis of sampled-data, high-gain control of interconnected systems. After stating the problem explicitly, open-loop behavior of subsystems are obtained based on the analysis in Chapter 2. Then, the rule of choosing the sampling intervals are mentioned before an explanatory example. Next, by applying the discrete dynamic output feedback controller, closed-loop behavior of the sampled system is obtained. Stabilization analysis is done based on the methodology in Chapter 2. Lastly, for unbounded systems, a proper adaptation action is proposed to obtain overall continuous-time stability.

In Chapter 4, an explanatory example of sampled-data control of interconnected systems are presented based on the method in Chapter 3. As an interconnected system, three coupled inverted penduli system is considered with a coupling spring connector. The stabilization methodology is applied to the system and the results are obtained with the help of a computer simulation.

Last Chapter is devoted to concluding remarks by revisiting the important points of the high-gain sampled-data control.

Chapter 2

A REVIEW OF HIGH-GAIN CONTROL

2.1 Two Canonical Forms

In this section, we present two canonical forms for single-input (single-input/ single-output) systems which we shall frequently refer to throughout the thesis, and at the same time introduce the notation used.

Consider a single-input system described as

$$\mathcal{S} : \dot{x}_p = A_p x_p + b_p u \tag{2.1}$$

where $x_p \in \Re^n$ is the state of $\mathcal{S}$, $u \in \Re$ is a scalar input, and A_p and b_p are constant matrices of appropriate dimensions. $\mathcal{S}$ can be denoted by the pair $\mathcal{S} = (A_p, b_p)$. It is well known that if $\mathcal{S}$ is controllable, then by a suitable coordinate transformation $x_p = Tx$ it can be transformed into an equivalent system $\mathcal{S} = (A, b)$, where

$$\begin{aligned} A &= T^{-1} A_p T = A_f + b_f d_f^T \\ b &= T^{-1} b_p = b_f \end{aligned} \tag{2.2}$$

with

$$A_f = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}; \quad b_f = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}; \quad d_f^T = \begin{bmatrix} -d_n & \dots & -d_1 \end{bmatrix} \tag{2.3}$$

The pair (A, b) is said to be in controllable canonical form. It is a useful structure in constructing stabilizing state feedback laws as we consider in the next section.

Now consider a single-input/single-output (SISO), controllable and observable system

$$\begin{aligned} \mathcal{S} : \dot{x}_p &= A_p x_p + b_p u \\ y &= c_p^T x_p \end{aligned} \tag{2.4}$$

which is represented by a triple $\mathcal{S} = (A_p, b_p, c_p^T)$. Let $\mathcal{S}$ have the scalar transfer function

$$h(s) = c_p^T (sI - A_p)^{-1} b_p = q_0 \frac{q(s)}{p(s)} = q_0 \frac{s^{n_o} + q_1 s^{n_o - 1} + \dots + q_{n_o}}{s^n + p_1 s^{n-1} + \dots + p_n} \tag{2.5}$$

$\mathcal{S}$ is said to have the relative degree

$$n_f = n - n_o = \deg(p) - \deg(q) \tag{2.6}$$

and the high-frequency gain q_o. If $\mathcal{S}$ is stable, $h(s)$ behaves like $h_f(s) = q_0 / s^{n_f}$ for large $\mid s \mid$. It has been shown [12] that $\mathcal{S} = (A_p, b_p, c_p^T)$ can be transformed into an equivalent system $\mathcal{S} = (A, b, c^T)$ with

$$\begin{aligned} A &= \begin{bmatrix} A_o & d_{of} c_f^T \\ b_f d_{fo}^T & A_f + b_f d_{ff}^T \end{bmatrix}, \quad b = q_0 \begin{bmatrix} 0 \\ b_f \end{bmatrix} \\ c^T &= \begin{bmatrix} 0 & c_f^T \end{bmatrix} \end{aligned} \tag{2.7}$$

where A_f and b_f have the structure in (2.3) with A_f being of order n_f and b_f of compatible size;

$$c_f^T = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

and A_o is of order $n_o = n - n_f$ and has the characteristic polynomial

$$det(sI - A_o) = q(s) = s^{n_o} + q_1 s^{n_o - 1} + \dots + q_{n_o} \tag{2.8}$$

2.2 High-gain State Feedback

Consider a system with nonlinear, time varying perturbations described as

$$\mathcal{S} : \dot{x} = Ax + bu + e(t, x) \tag{2.9}$$

where we assume that the nominal system (A, b) is controllable and the perturbations satisfy the matching conditions [6]

$$e(t, x) = bg(t, x) \tag{2.10}$$

We further assume that g in (2.10) is bounded as

$$\mid g(t, x) \mid \leq \alpha_g \parallel x \parallel \tag{2.11}$$

for some $\alpha_g > 0$. Without loss of generality, assume that the pair (A, b) is already transformed into its controllable canonical form in (2.2) with the term $b_f d_f^T x$ included in the perturbation; that is, assume

$$A = A_f, \quad b = b_f$$

where A_f and b_f are as in (2.3).

To stabilize $\mathcal{S}$, we use a state feedback control

$$u = -k^T x, \quad k^T = \begin{bmatrix} k_n & k_{n-1} & \dots & k_1 \end{bmatrix} \tag{2.12}$$

which results in a closed-loop system

$$\hat{\mathcal{S}} : \dot{x} = \hat{A}_f x + b_f g(t, x) \tag{2.13}$$

where

$$\hat{A}_f = A_f - b_f k^T = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -k_n & -k_{n-1} & \dots & -k_1 \end{bmatrix}$$

$\hat{A}_f$ is in companion form with the characteristic polynomial

$$\hat{d}(s) = s^n + k_1 s^{n-1} + \cdots + k_n \tag{2.14}$$

Let k^T be chosen such that $\hat{A}_f$ has distinct eigenvalues

$$\lambda_i = -\rho\sigma_i, \quad i = 1, 2, \ldots, n \tag{2.15}$$

where $\sigma_i > 0$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and $\rho > 0$ is a parameter to be specified. It is known that $\hat{A}_f$ has a modal matrix

$$\begin{aligned}
\hat{Q} &= \begin{bmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ & \rho & & \\ & & \ddots & \\ & & & \rho^{n-1} \end{bmatrix} \begin{bmatrix} 1 & \ldots & 1 \\ -\sigma_1 & \ldots & -\sigma_n \\ \vdots & & \vdots \\ (-\sigma_1)^{n-1} & \ldots & (-\sigma_n)^{n-1} \end{bmatrix} = RQ
\end{aligned} \tag{2.16}$$

such that

$$\hat{Q}^{-1}\hat{A}_f\hat{Q} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = -\rho \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} = -\rho D$$

The transformation $x = \hat{Q}\hat{x}$, transforms the closed-loop system $\hat{\mathcal{S}}$ into

$$\hat{\mathcal{S}} : \dot{\hat{x}} = -\rho D\hat{x} + \hat{e}(t, \hat{x}) \tag{2.17}$$

where

$$\begin{aligned}
\hat{e}(t, \hat{x}) &= \hat{Q}^{-1} b_f g(t, \hat{Q}\hat{x}) \\
&= Q^{-1} R^{-1} b_f g(t, \hat{Q}\hat{x}) \\
&= Q^{-1} \rho^{1-n} b_f g(t, \hat{Q}\hat{x}) \\
&= \rho^{1-n} Q^{-1} b_f g(t, \hat{Q}\hat{x})
\end{aligned} \tag{2.18}$$

Clearly,

$$
\begin{aligned}
\| \hat{e}(t,\hat{x}) \| &\leq \rho^{1-n} \| Q^{-1}b_f \| | g(t,\hat{Q}\hat{x}) | \\
&\leq \alpha_g \rho^{1-n} \| Q^{-1}b_f \| \| \hat{Q}\hat{x} \| \\
&\leq \alpha_g \rho^{1-n} \| Q^{-1}b_f \| \| R \| \| Q \| \| \hat{x} \| \\
&\leq \alpha_g \| Q^{-1}b_f \| \| Q \| \| \hat{x} \| \\
&\leq \hat{\alpha}_g \| \hat{x} \| \qquad (2.19)
\end{aligned}
$$

and $\hat{\alpha}_g$ is independent of the gain parameter ρ.

Let $v(\hat{x}) = \| \hat{x} \|^2 = \hat{x}^T\hat{x}$ be a candidate for a Lyapunov function for $\hat{\mathcal{S}}$. Then

$$
\begin{aligned}
\dot{\hat{v}} &= 2\hat{x}^T\big(-\rho D\hat{x} + \hat{e}(t,\hat{x})\big) \\
&\leq -2(\sigma_{\min}\rho - \hat{\alpha}_g) \| \hat{x} \|^2 \qquad (2.20)
\end{aligned}
$$

Whatever $\hat{\alpha}_g$ is, for a given $\sigma > 0$, ρ can be chosen sufficiently large to have $\sigma_{\min}\rho - \hat{\alpha}_g \geq \sigma$ so that $\dot{v}(\hat{x}) \leq -2\sigma v(\hat{x})$. This shows that the closed-loop system can be made exponentially stable with arbitrary degree σ of stability.

Note that the closed-loop characteristic polynomial is of the form

$$\hat{d}(s) = s^n + \rho d_1 s^{n-1} + \cdots + \rho^n d_n \qquad (2.21)$$

where $d_1, \ldots, d_n$ are uniquely determined by $\sigma_1, \ldots, \sigma_n$ and are fixed. Comparing (2.21) and (2.14), we observe that

$$k^T = \begin{bmatrix} \rho^n d_n & \rho^{n-1} d_{n-1} & \ldots & \rho d_1 \end{bmatrix}$$

that is, the stabilizing control in (2.12) is a high-gain state feedback.

2.3 High-gain Dynamic Output Feedback

Consider a single-input/single-output (SISO) system with nonlinear, time varying perturbations

$$
\begin{aligned}
\mathcal{S} : \dot{x} &= Ax + bu + e(t,x) \\
y &= c^T x
\end{aligned} \tag{2.22}
$$

where $y \in \Re$ is the scalar output of the system. We assume that the perturbations are of the form

$$e(t,x) = bg(t,x) + h(t,y) \tag{2.23}$$

Note that the first term $bg(t,x)$ in (2.23) satisfies the matching condition on the input side and the second term $h(t,y) = h(t, c^T x)$ on the output side. We further assume that g is bounded as in (2.11) and h is bounded as

$$\| h(t,y) \| \leq \alpha_h \mid y \mid \tag{2.24}$$

for some $\alpha_h > 0$.

We also make the following assumptions concerning the nominal system (A, b, c^T).

- (A, b, c^T) is controllable and observable
- (A, b, c^T) has stable zeros, that is, $q(s)$ in (2.5) is stable.
- the relative degree $n_f = n - n_o$ and the high-frequency gain q_o are known.

We assume without loss of generality that A, b and c^T are already transformed into the forms in (2.7). Then including the $b_f d_{fo}^T x_o$ and $b_f d_{ff}^T x_f$ terms in $bg(t,x)$ and $d_{of} c_f^T x_f$ term in $h(t,y)$, the system in (2.22) can be described as

$$
\begin{aligned}
S : \dot{x}_o &= A_o x_o + h_o(t,y) \\
\dot{x}_f &= A_f x_f + q_0 b_f u + b_f g(t, x_0, x_f) + h_f(t,y) \\
y &= c_f^T x_f
\end{aligned} \tag{2.25}
$$

To the system $\mathcal{S}$, we apply a dynamic output feedback control of the form [18]

$$\begin{aligned}\mathcal{C} : \dot{x}_c &= \rho A_c x_c + \rho^{n_f-1} b_c y \\ u &= q_0^{-1}\left(\rho c_c^T x_c + \rho^{n_f-1} d_c y\right)\end{aligned} \tag{2.26}$$

where $x_c \in \Re^{n_f-1}$, ρ is a gain parameter to be specified and A_c, b_c, c_c^T, and d_c are constant matrices such that

$$\hat{A}_f = \begin{bmatrix} A_f + b_f d_c c_f^T & b_f c_c^T \\ b_c c_f^T & A_c \end{bmatrix} \tag{2.27}$$

is stable [2].

Defining

$$\hat{x}_o = x_o, \quad \hat{x}_f = \begin{bmatrix} R_f^{-1} x_f \\ x_c \end{bmatrix} \tag{2.28}$$

where

$$R_f = \begin{bmatrix} 1 & & & \\ & \rho & & \\ & & \ddots & \\ & & & \rho^{n_f-1} \end{bmatrix}$$

and noting that

$$R_f^{-1} A_f R_f = \rho A_f, \quad R_f^{-1} b_f = \rho^{1-n_f} b_f, \quad c_f^T R_f = c_f^T \tag{2.29}$$

the closed-loop system $\hat{\mathcal{S}}$ consisting of $\mathcal{S}$ and $\mathcal{C}$ is described by

$$\begin{aligned}\hat{\mathcal{S}} : \dot{\hat{x}}_o &= A_o \hat{x}_o + \hat{e}_o(t, \hat{x}_o, \hat{x}_f) \\ \dot{\hat{x}}_f &= \rho \hat{A}_f \hat{x}_f + \hat{e}_f(t, \hat{x}_o, \hat{x}_f)\end{aligned} \tag{2.30}$$

Although we have included $b_f d_{fo}^T x_o$ and $b_f d_{ff}^T x_f$ in $bg(t,x)$ and $d_{of} c_f^T x_f$ term in $h(t,y)$, we state these terms explicitly here to see their effects on the perturbations:

$$\begin{aligned}
\hat{e}_o(t,\hat{x}_o,\hat{x}_f) &= d_{of} c_f^T x_f + h_o(t, c_f^T x_f) \\
\hat{e}_f(t,\hat{x}_o,\hat{x}_f) &= \begin{bmatrix} \hat{e}_{f1}(t,\hat{x}_o,\hat{x}_f) \\ 0 \end{bmatrix} \\
\hat{e}_{f1}(t,\hat{x}_o,\hat{x}_f) &= R_f^{-1} b_f d_{fo}^T x_o + R_f^{-1} b_f d_{ff}^T x_f + q_0 R_f^{-1} b_f g(t, x_o, x_f) \\
&+ R_f^{-1} h_f(t, c_f^T x_f)
\end{aligned} \tag{2.31}$$

It is not difficult to show using (2.23), (2.24) and (2.29)that

$$\begin{aligned}
\| \hat{e}_o(t,\hat{x}_o,\hat{x}_f) \| &\leq \alpha_{of} \| \hat{x}_f \| \\
\| \hat{e}_f(t,\hat{x}_o,\hat{x}_f) \| &\leq \alpha_{fo} \| \hat{x}_o \| + \alpha_{ff} \| \hat{x}_f \|
\end{aligned}$$

for some α_{of}, α_{fo} and $\alpha_{ff} > 0$.

Since A_o is stable by assumption and $\hat{A}_f$ is made stable by the choice of the controller parameters, there exist positive definite matrices P_o and P_f such that

$$\begin{aligned}
A_o^T P_o + P_o A_o &= -I \\
\hat{A}_f^T P_f + P_f \hat{A}_f &= -I
\end{aligned} \tag{2.32}$$

We now choose $v(\hat{x}_o,\hat{x}_f) = \hat{x}_o^T P_o \hat{x}_o + \hat{x}_f^T P_f \hat{x}_f$ as a Lyapunov Function for $\hat{\mathcal{S}}$. Using (2.30), (2.32) and (2.32), $\dot{v}$ can be majorized as

$$\dot{v}(\hat{x}_o,\hat{x}_f) \leq -\xi^T Q(\rho) \xi \tag{2.33}$$

where

$$\begin{aligned}
\xi &= \begin{bmatrix} \| \hat{x}_o \| & \| \hat{x}_f \| \end{bmatrix}^T \quad \textit{and} \\
Q(\rho) &= \begin{bmatrix} 1 & -\alpha_{of} \| P_o \| - \alpha_{fo} \| P_f \| \\ -\alpha_{of} \| P_o \| - \alpha_{fo} \| P_f \| & \rho - 2\alpha_{ff} \| P_f \| \end{bmatrix}
\end{aligned}$$

From (2.33), we observe that for given bounds α_{of}, α_{fo} and α_{ff} and any given $0 < \alpha < 1$, ρ can be chosen sufficiently large to have $\lambda_{min}(Q) \geq \alpha$, so that

$$\dot{v}(\hat{x}_o, \hat{x}_f) \leq -\alpha \parallel \xi \parallel^2 \leq -2\sigma v(\hat{x}_o, \hat{x}_f) \tag{2.34}$$

where

$$\sigma = \frac{1}{2} \frac{\alpha}{\max\left\{\lambda_{max}(P_o), \lambda_{max}(P_f)\right\}}$$

This shows that $\hat{\mathcal{S}}$ can be made exponentially stable with degree of stability σ, which depends mainly on the degree of stability of A_o and the perturbation bounds.

The argument above is valid even when the gain ρ is time-varying provided that $\dot{\rho}$ is bounded. Boundedness of $\mid \dot{\rho} \mid$ is required because of the fact that when ρ is time-varying then the transformation in (2.28) introduces additional perturbation terms (containing $\dot{\rho}$) into the closed-loop system $\hat{\mathcal{S}}$ in (2.30). However, as long as $\mid \dot{\rho} \mid$ is bounded, say $\mid \dot{\rho} \mid \leq 1$, then there exists a critical value $\rho = \rho^\star$ for which $Q(\rho^\star)$ in (2.33) (actually a modified version of it that also takes into account the effect of $\mid \dot{\rho} \mid$) is just positive-definite, so that $\hat{\mathcal{S}}$ is stable for any $\rho > \rho^\star$. Clearly, $\rho^\star$ depends on the perturbation bounds (as well as the nominal closed-loop system parameters). If these bounds are not known, then ρ must be adjusted by an adaptation mechanism which increases ρ (slowly) to a sufficiently high (but bounded) value for which $\hat{\mathcal{S}}$ is stable. Based on this observation, the gain-adaptation rule is chosen as [8]

$$\dot{\rho}(t) = \min\left\{1, \alpha_y \mid y \mid^2 + \alpha_z \parallel x_c \parallel^2 \right\} \tag{2.35}$$

where $\alpha_y > 0$ and $\alpha_z > 0$ are arbitrary constants.

The adaptation mechanism works as follows. If $\rho(t) < \rho^\star$ for all $t \geq t_0$, then (2.35) implies that $\rho(t) \longrightarrow \rho_\infty \leq \rho^\star$, which in turn requires that $y(t) \longrightarrow 0$ and $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Then, by (2.26), we have $u(t) \longrightarrow 0$ and controllability and observability of the nominal system implies that $x(t) \longrightarrow 0$ as $t \longrightarrow 0$.

On the other hand, if $\rho(t^\star) \geq \rho^\star$ for some $t^\star \geq t_0$, then $\hat{\mathcal{S}}$ is exponentially stable, so that

$$| y(t) | \leq M_y e^{-\sigma(t-t^\star)} | y(t^\star) |$$

and

$$| z(t) | \leq M_z e^{-\sigma(t-t^\star)} | z(t^\star) |$$

for $t > t^\star$, where σ is the degree of exponential stability of $\hat{\mathcal{S}}$. Then, from (2.35), we obtain

$$\begin{aligned} \rho(t) &\leq \rho(t^\star) + \int_{t^\star}^{t} (\alpha_y | y(\tau) |^2 + \alpha_z \| z(\tau) \|^2) d\tau \\ &\leq \rho(t^\star) + M_\rho [1 - e^{-2\sigma(t-t^\star)}] \end{aligned}$$

where

$$M_\rho = \frac{\alpha_y M_y^2 | y(t^\star) |^2 + \alpha_z M_z^2 \| z(t^\star) \|^2}{2\sigma}$$

Hence, $\rho(t) \longrightarrow \rho_\infty = \rho(t^\star) + M_\rho$ as $t \longrightarrow \infty$. This shows that the adaptation rule in (2.35) does not result in an ever-increasing gain.

2.4 Sampled-data Output Feedback Control

Once it is shown that the perturbed system $\mathcal{S}$ in (2.22) can be stabilized by a high-gain dynamic output feedback controller $\mathcal{C}$ as in (2.26), a natural question is whether $\mathcal{S}$ can be stabilized by a discrete version of $\mathcal{C}$ operating on sampled values of the output.

Let $t_k, \quad k = 0, 1, \ldots,$ denote the sampling instants and let $T_k = t_{k+1} - t_k$ denote the sampling intervals. To provide maximum flexibility in the analysis, we consider a non-uniform sampling, that is, T_k is not necessarily a constant.

Letting $t = t_k + sT_k, \quad 0 \le s \le 1$, defining

$$\begin{aligned} x_{ok}(s) &= x_o(t_k + sT_k) \\ x_{fk}(s) &= D_{fk}^{-1} x_f(t_k + sT_k) \\ u_k(s) &= u(t_k + sT_k) \\ y_k(s) &= y(t_k + sT_k) \end{aligned} \tag{2.36}$$

where

$$D_{fk} = \begin{bmatrix} T_k^{n_f - 1} & & & \\ & \ddots & & \\ & & T_k & \\ & & & 1 \end{bmatrix} \tag{2.37}$$

and noting that

$$\begin{aligned} D_{fk}^{-1} A_f D_{fk} &= T_k^{-1} A_f \\ D_{fk}^{-1} b_f &= b_f \\ c_f^T D_{fk} &= T_k^{n_f - 1} c_f \end{aligned}$$

the behavior of $\mathcal{S}$ in (2.22) over the k-th sampling interval can be described by

$$\begin{aligned} \mathcal{S} : \dot{x}_{ok}(s) &= T_k A_o x_{ok}(s) + T_k e_{ok}\big(s, x_{fk}(s)\big) \\ \dot{x}_{fk}(s) &= A_f x_{fk}(s) + T_k e_{fk}\big(s, x_{ok}(s), x_{fk}(s)\big) + q_0 T_k b_f u_k(s) \\ y_k(s) &= T_k^{n_f - 1} c_f^T x_{fk}(s) \end{aligned} \tag{2.38}$$

where the perturbations e_{ok} and e_{fk} satisfy

$$\begin{aligned} \| \, e_{ok}(s, x_{fk}) \, \| &\le \alpha_{of} \, \| \, x_{fk} \, \| \\ \| \, e_{fk}(s, x_{ok}, x_{fk}) \, \| &\le \alpha_{fo} \, \| \, x_{ok} \, \| + \alpha_{ff} \, \| \, x_{fk} \, \| \end{aligned} \tag{2.39}$$

for some α_{of}, α_{fo} and $\alpha_{ff} > 0$.

The controller to be used for stabilization of the perturbed system in (2.38) is a discrete version of $\mathcal{C}$ in (2.26). Observing that a faithful discretization of a high-gain controller requires fast sampling, $\rho_k = T_k^{-1}$ seems to be a reasonable choice for the controller gain. This choice has the additional advantage of providing simplicity in the stability analysis as only a single parameter, T_k, is used to adjust both the sampling interval and the controller gain. Based on this observation the following sampled-data controller is proposed [8]

$$\begin{aligned} \mathcal{C}_D : x_c[k+1] &= A_c x_c[k] + T_k^{1-n_f} b_c y(t_k) \\ w[k] &= T_k^{-1} c_c^T x_c[k] + T_k^{-n_f} d_c y(t_k) \\ u_k(s) &= q_o^{-1} w[k], \quad 0 \leq s < 1 \end{aligned} \tag{2.40}$$

where $x_c[k] \in \Re^{n_f - 1}$ is the discrete state of $\mathcal{C}_D$ at $t = t_k$.

As shown in [8], the behavior of the closed-loop system consisting of $\mathcal{S}$ in (2.38) and the controller $\mathcal{C}_D$ in (2.40) at the sampling instants can be described by a discrete model

$$\begin{aligned} \hat{\mathcal{S}}_D : \hat{x}_o[k+1] &= \hat{\Phi}_o \hat{x}_o[k] + \hat{\xi}_{ok}\big(k, \hat{x}_o[k], \hat{x}_f[k]\big) \\ \hat{x}_f[k+1] &= \hat{\Phi}_f \hat{x}_f[k] + \hat{\xi}_{fk}\big(k, \hat{x}_o[k], \hat{x}_f[k]\big) \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} \hat{x}_o[k] &= x_{ok}(0) \\ \hat{x}_f[k] &= \begin{bmatrix} x_{fk}(0) \\ x_c[k] \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \hat{\Phi}_o &= e^{T_k A_o} \\ \hat{\Phi}_f &= \begin{bmatrix} e^{A_f} + \Gamma_f d_c c_f^T & \Gamma_f c_c^T \\ b_c c_f^T & A_c \end{bmatrix} \end{aligned} \tag{2.42}$$

with

$$\Gamma_f = \int_0^1 e^{A_f \tau} b_f d\tau$$

It is further shown in [8] that if the sampling intervals T_k are such that

$$\begin{aligned} T_{k+1} &\leq T_k < 1 \\ (\frac{T_k}{T_{k+1}})^{n_f - 1} &\leq 1 + T_k \end{aligned} \tag{2.43}$$

then the perturbation terms in (2.41) can be bounded as

$$\begin{aligned} \| \hat{\xi}_{ok}(k, \hat{x}_o, \hat{x}_f) \| &\leq T_k^2 \beta_{oo} \| \hat{x}_o \| + T_k \beta_{of} \| \hat{x}_f \| \\ \| \hat{\xi}_{fk}(k, \hat{x}_o, \hat{x}_f) \| &\leq T_k \beta_{fo} \| \hat{x}_o \| + T_k \beta_{ff} \| \hat{x}_f \| \end{aligned} \tag{2.44}$$

Since A_o is assumed to be stable, there exists a positive definite matrix P_o such that

$$A_o^T P_o + P_o A_o = -I$$

which implies that

$$\| \hat{\Phi}_o^T P_o \hat{\Phi}_o - P_o \| \leq -\gamma_o T_k \tag{2.45}$$

for some $\gamma_o > 0$. Also, $\hat{\Phi}_f$ in (2.42) represents the system matrix of a controllable and observable discrete system $(e^{A_f}, \Gamma_f, c_f^T)$ in a feedback configuration with a discrete controller (A_c, b_c, c_c^T, d_c), and thus can be made Shur-stable by a suitable choice of the controller parameters [2]. Then there exists a positive definite P_f such that

$$\hat{\Phi}_f^T P_f \hat{\Phi}_f - P_f = -I \tag{2.46}$$

Let $v[k] = \hat{x}_o^T[k] P_o \hat{x}_o[k] + \hat{x}_f^T[k] P_f \hat{x}_f[k]$ be a candidate for a Lyapunov Function for the discrete closed-loop system in (2.41). Then, (2.43)-(2.46) imply that there exists a $T_\star < 1$ that depends on the perturbation bounds such that the difference of $v[k]$ along the solutions of $\hat{\mathcal{S}}_D$ can be bounded as

$$\Delta v[k] \leq -\sigma T_k v[k] \tag{2.47}$$

for some $\sigma > 0$ provided $T_k \leq T_\star$. This shows that the discrete model of the closed-loop system can be made exponentially stable by means of a discrete controller having a sufficiently high gain and operating on sufficiently frequent output samples. If T_k is also bounded from below so that $t_k = t_0 + \sum_{j=0}^{k-1} T_j \longrightarrow \infty$ as $k \longrightarrow \infty$, then the closed-loop sampled-data system $\hat{\mathcal{S}}$ is also stable.

As in the continuous-time case, if the perturbation bounds are not known, then T_k is adjusted by an adaptation rule

$$T_{k+1}^{-1} = T_k^{-1} + T_k \min \left\{ \alpha_o, \alpha_y \mid y(t_k) \mid^2 + \alpha_z \parallel z(k) \parallel^2 \right\} \tag{2.48}$$

where

$$\alpha_o = 2^{\frac{1}{n_f - 1}} - 1$$

and $\alpha_y > 0$ and $\alpha_z > 0$ arbitrary. This not only guarantees the restrictions in (2.43), but also the requirement that

$$\lim_{k \to \infty} T_k = T_\infty > 0$$

2.5 Decentralized Control of Interconnected Systems

A natural extension of high-gain stabilization technique considered in the previous sections is decentralized control of interconnected system that consist of N subsystems described as

$$\begin{aligned} \mathcal{S}_i : \dot{x}_i &= A_i x_i + b_i u_i + e_i(t, x) \\ y_i &= c_i^T x_i \end{aligned} \tag{2.49}$$

where $x_i(t) \in \Re^{n_i}$ is the state of $\mathcal{S}_i$ $u_i(t) \in \Re$ and $y_i(t) \in \Re$ are scalar input and output of $\mathcal{S}_i$, and $e_i(t, x)$ represents the interconnections between $\mathcal{S}_i$ and other subsystems with

$$x = \left[\begin{array}{cccc} x_1^T & x_2^T & \ldots & x_N^T \end{array} \right]^T = \mathrm{col}\,[x_i]$$

It is observed that the interconnections can be treated as perturbations on the nominal subsystems described by the triplets (A_i, b_i, c_i^T).

As in the case of a single system, we assume that

- (A_i, b_i, c_i^T) are controllable and observable

• with $h_i(s) = c_i^T(sI - A_i)^{-1}b_i = q_{0i}\frac{q_i(s)}{p_i(s)}$, the zeros of $q_i(s)$ are stable

• high-frequency gain q_{0i} and the relative degree $n_{fi} = \deg(p_i) - \deg(q_i)$ of each subsystem are known

• the interconnection terms are of the form

$$e_i(t, x) = b_i g_i(t, x) + h_i(t, y) \tag{2.50}$$

where $y = \mathrm{col}\,[y_i]$, and

$$\begin{aligned} \mid g_i(t, x) \mid &\leq \sum_{j=1}^{N} \alpha_{ij}^g \parallel x_j \parallel \\ \parallel h_i(t, y) \parallel &\leq \sum_{j=1}^{N} \alpha_{ij}^h \mid y_j \mid \end{aligned} \tag{2.51}$$

for some constants α_{ij}^g, $\alpha_{ij}^h > 0$.

The overall system can be represented as

$$\begin{aligned} \mathcal{S} : \dot{x} &= Ax + Bu + E(t, x) \\ y &= Cx \end{aligned}$$

with obvious definitions of x, u, y and A, B, C and E. The assumptions on (A, B, C) and the perturbations $E(t, x)$ allows for the design of a centralized high-gain dynamic output feedback controller that stabilizes $\mathcal{S}$. As shown in [18], stability can also be achieved by means of decentralized output-feedback controllers provided their gains are in certain proportions that depend on the relative degrees of the subsystems. In other words, the local controller for the i-th subsystem is chosen as

$$\begin{aligned} \mathcal{C}_i : \dot{x}_{ci} &= \rho_i A_c x_{ci} + \rho_i^{n_{fi}-1} b_c y_i \\ u_i &= \rho_i c_{ci}^T x_{ci} + \rho_i^{n_{fi}-1} d_{ci} y_i \end{aligned} \tag{2.52}$$

where local gains are generated from a common gain as

$$\rho_i = \rho^{\nu_i} \tag{2.53}$$

where ν_i depend (in a complicated way) on the relative degrees of the subsystems.

It has been shown in [18] by a Lyapunov analysis that the overall system in (2.49) can be stabilized by means of decentralized controllers in (2.52) provided ρ is sufficiently high. As discussed in Section 2.3, ρ can even be time-varying as long as $\dot{\rho}$ is bounded. As in the case of a single system, how high ρ should be depends on the bounds of the strength of interconnections. If these bounds are not known, then it can be adjusted by a centralized adaptation rule

$$\dot{\rho} = \min\left\{1, \alpha_y \parallel y \parallel^2 + \alpha_z \parallel x_c \parallel^2 \right\} \tag{2.54}$$

where $x_c = \text{col}\,[x_{ci}]$.

The main difficulty arises when we consider stabilization of the interconnected system in (2.49) by means of decentralized sampled-data controllers. This problem is the main topic of the thesis and is considered in the next chapter.

Chapter 3

DECENTRALIZED SAMPLED-DATA CONTROL

3.1 Problem Statement

Consider an interconnected system consisting of N subsystems $\mathcal{S}_i$ described in (2.49). Under the assumptions mentioned in Section 2.5, we transform each subsystem to the canonical form in (2.25) and describe it as

$$\begin{aligned} \mathcal{S}_i : \dot{x}_{oi}(t) &= A_{oi}x_{oi}(t) + e_{oi}\big(t, x_f(t)\big) \\ \dot{x}_{fi}(t) &= A_{fi}x_{fi}(t) + e_{fi}\big(t, x_o(t), x_f(t)\big) + q_{0i}b_{fi}u_i(t) \\ y_i(t) &= c_{fi}^T x_{fi}(t) \end{aligned} \tag{3.1}$$

where $x_{oi} \in \Re^{n_{oi}}$, $x_{fi} \in \Re^{n_{fi}}$, $u_i \in \Re$, $y_i \in \Re$ and

$$x_o = \mathrm{col}\,[x_{oi}], \quad x_f = \mathrm{col}\,[x_{fi}], \quad y = \mathrm{col}\,[y_i]$$

We also assume that the interconnections also satisfy the conditions in Section 2.5, that is

$$\begin{aligned} e_{oi}(t, x_f) &= h_{oi}(t, y) \\ e_{fi}(t, x_o, x_f) &= b_{fi}g_i(t, x) + h_{fi}(t, y) \end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
\mid g_i(t,x) \mid \ &\leq \ \left(\sum_{j=1}^{N} \alpha_{ij}^{go} \parallel x_{oj} \parallel + \alpha_{ij}^{gf} \parallel x_{fj} \parallel \right) \\
\parallel h_{oi}(t,x_f) \parallel \ &\leq \ \sum_{j=1}^{N} \alpha_{ij}^{ho} \mid c_{fj}^T x_{fj} \mid \\
\parallel h_{fi}(t,x_f) \parallel \ &\leq \ \sum_{j=1}^{N} \alpha_{ij}^{hf} \mid c_{fj}^T x_{fj} \mid
\end{aligned}
\tag{3.3}
$$

for some $\alpha_{ij}^{go} > 0$, $\alpha_{ij}^{gf} > 0$, $\alpha_{ij}^{ho} > 0$ and $\alpha_{ij}^{hf} > 0$ with $i,j \in 1,\ldots,N$.

Our purpose is to stabilize the overall interconnected system by using discrete version of the decentralized controllers in (2.52) operating on sampled values of local outputs. To guarantee synchronous operation of the controllers, which is needed to derive a discrete-time model of the closed-loop system, we assume that each output is sampled an integer number of times in a certain common sampling interval. That is, if

$$T_k = t_{k+1} - t_k \tag{3.4}$$

denote the k-th common sampling interval, the i-th controller takes uniform samples of $y_i(t)$ separated by

$$T_{ik} = \frac{T_k}{M_{ik}} \tag{3.5}$$

where M_{ik} is an integer, Note that the common sampling interval is not constant; in fact, it is deliberately assumed to be non-constant to allow for adaptive adjustment. Similarly, the number of samples taken by the i-th controller in a common sampling interval is not constant, although samples are uniform throughout each common sampling interval.

We now turn our attention to the process of discretizing local controllers in (2.52). To provide simplicity in the design of the controllers, we set the gain of each controller to the reciprocal of its sampling interval, as we did in Section 2.4, that is

$$\rho_i(t) = T_{ik}^{-1}, \quad t_k \leq t < t_{k+1} \tag{3.6}$$

Recall, however, that to achieve stability of the overall system with decentralized control, gains of the controllers are required to be in certain proportions; that is

$$\rho_i(t) = \rho^{\nu_i}(t) \tag{3.7}$$

where $\nu_i > 0$ are integers that depend on the relative degrees of the subsystems. In terms of T_{ik}, (3.7) requires

$$T_{ik} = \rho_k^{-\nu_i} \tag{3.8}$$

for some $\rho_k > 0$. To satisfy (3.5) and (3.8) simultaneously, we choose $\rho_k = I_k \geq 1$, an integer. Then, with

$$T_{ik} = \frac{1}{I_k^{\nu_i}}, \quad i = 1, 2, \ldots, N \tag{3.9}$$

and

$$T_k = \frac{1}{I_k^{\nu_{\min}}} \tag{3.10}$$

where $\nu_{min} = \min\{\nu_i\}$, we observe that

$$T_k = I_k^{\nu_i - \nu_{\min}} T_{ik} = M_{ik} T_{ik} \tag{3.11}$$

that is, (3.5) is also satisfied

Finally, we define the largest common measure of T_{ik}'s as the basic unit interval in the k-th common sampling interval and denote it by τ_k. Thus

$$\tau_k = \frac{1}{I_k^{\nu_{\max}}} \tag{3.12}$$

where $\nu_{\max} = \max\{\nu_i\}$. Clearly, each local sampling interval T_{ik} contains an integral number of τ_k, that is

$$T_{ik} = I_k^{\nu_{\max} - \nu_i} \tau_k = N_{ik} \tau_k \tag{3.13}$$

Note that

$$M_{ik} N_{ik} = I_k^{\nu_{\max} - \nu_{\min}} = L_k, \quad i = 1, 2, \ldots, N \tag{3.14}$$

so that

$$T_k = L_k \tau_k \tag{3.15}$$

3.2 Open-Loop Behavior of The Interconnected System and Sample-Rate Selection

As a first step to derive a discrete-time model for the closed-loop interconnected system we obtain expressions for the solutions of the subsystems with $u_i(t)$ in (3.1) as external inputs supplied by local sampled-data controllers. Since τ_k is the largest interval over which all $u_i(t)$ are constant, we analyze the behavior of the subsystems over each interval

$$t_k + l\tau_k \leq t \leq t_k + (l+1)\tau_k, \quad l = 0, 1, \ldots, L_k - 1 \tag{3.16}$$

separately. For this purpose, we let $t = t_k + l\tau_k + s\tau_k$, $0 \leq s \leq 1$, and define

$$\begin{aligned} x_{oikl}(s) &= x_{oi}(t_k + l\tau_k + s\tau_k) \\ x_{fikl}(s) &= D_{fik}^{-1} x_{fi}(t_k + l\tau_k + s\tau_k) \end{aligned} \tag{3.17}$$

where

$$D_{fik} = \begin{bmatrix} T_{ik}^{m_i - 1} & & & \\ & \ddots & & \\ & & T_{ik} & \\ & & & 1 \end{bmatrix} \tag{3.18}$$

with $m_i = n_{fi}$ for simplicity in notation.

On noting that

$$\begin{aligned} \tau_k D_{fik}^{-1} A_{fi} D_{fik} &= \frac{\tau_k}{T_{ik}} A_{fi} = \frac{1}{N_{ik}} A_{fi} = A_{fik} \\ D_{fik}^{-1} b_{fi} &= b_{fi} \\ c_{fi}^T D_{fik} &= T_{ik}^{m_i - 1} c_{fi}^T \end{aligned} \tag{3.19}$$

and defining the auxiliary variable w_{ikl} as

$$w_{ikl} = q_{0i} u_i(t), \quad t_k + l\tau_k \leq t < t_k + (l+1)\tau_k \tag{3.20}$$

subsystem descriptions in (3.1) are transformed into

$$
\begin{aligned}
\mathcal{S}_i : \dot{x}_{oikl}(s) &= \tau_k A_{oi} x_{oikl}(s) + e_{oikl}\big(s, x_{fkl}(s)\big) \\
\dot{x}_{fikl}(s) &= A_{fik} x_{fikl}(s) + e_{fikl}\big(s, x_{kl}(s)\big) + \tau_k b_{fi} v_{ikl} \\
y_{ikl}(s) &= T_{ik}^{m_i-1} c_{fi}^T x_{fikl}(s)
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
e_{oikl}\big(s, x_{fkl}(s)\big) &= \tau_k h_{oi}\big(t_k + l\tau_k + s\tau_k, D_{fk} x_{fkl}(s)\big) \\
e_{fikl}\big(s, x_{kl}(s)\big) &= \tau_k b_{fi} g_i\big(t_k + l\tau_k + s\tau_k, x_{okl}(s), D_{fk} x_{fkl}(s)\big) \\
&+ \tau_k D_{fik}^{-1} h_{fi}\big(t_k + l\tau_k + s\tau_k, C_f D_{fk} x_{fkl}(s)\big)
\end{aligned}
\tag{3.22}
$$

with $x_{okl} = \mathrm{col}\,[x_{oikl}]$, $x_{fkl} = \mathrm{col}\,[x_{fikl}]$, $C_f = \mathrm{diag}\,[c_{fi}^T]$ and $D_{fk} = \mathrm{diag}\,[D_{fik}]$.

Using (3.3), the interconnection terms in (3.22) can be bounded for $T_k \leq 1$ as

$$
\begin{aligned}
\| \, e_{oikl}(s, x_{fkl}) \, \| &\leq \tau_k \sum_{j=1}^{N} \alpha_{ij}^{ho} T_{jk}^{m_j-1} \, \| \, x_{fjkl} \, \| \\
\| \, e_{fikl}(s, x_{fkl}) \, \| &\leq \tau_k \sum_{j=1}^{N} (\alpha_{ij}^{go} \, \| \, x_{ojkl} \, \| + \alpha_{ij}^{gf} T_{jk}^{m_j-1} \, \| \, x_{fjkl} \, \|) \\
&+ \tau_k T_{ik}^{1-m_i} \sum_{j=1}^{N} \alpha_{ij}^{hf} T_{jk}^{m_j-1} \, \| \, x_{fjkl} \, \|
\end{aligned}
\tag{3.23}
$$

The key to stabilization of the interconnected system is to choose the local sampling intervals so as to have the smallest possible bounds on the interconnection in (3.23). For this purpose, we choose the integers ν_i in (3.9) as

$$
\nu_i = \begin{cases} \frac{\nu}{m_i - 1} & m_i \neq 1 \\ \nu + 1 & m_i = 1 \end{cases}
\tag{3.24}
$$

where

$$
\nu = \prod_{\substack{m_i \neq 1 \\ m_i \text{distinct}}} (m_i - 1)
\tag{3.25}
$$

	$\mathcal{O}(\alpha_{ij}^{of})$	$\mathcal{O}(\alpha_{ij}^{fo})$	$\mathcal{O}(\alpha_{ij}^{ff})$
$m_i = 1,\, m_j = 1$	ν_{max}	ν_{max}	ν_{max}
$m_i = 1,\, m_j \neq 1$	$\nu_{max} + \nu$	ν_{max}	$\nu_{max} + \nu$
$m_i \neq 1,\, m_j = 1$	ν_{max}	ν_{max}	$\nu_{max} - \nu$
$m_i \neq 1,\, m_j \neq 1$	$\nu_{max} + \nu$	ν_{max}	ν_{max}

Table 3.1: Orders of α_{ij}^{of}, α_{ij}^{fo}, and α_{ij}^{ff}

With this choice of $\nu_i's$, the bounds in (3.23) can be expressed as

$$\begin{aligned} \| e_{oikl}(s, x_{fkl}) \| &\leq \sum_{j=1}^{N} \alpha_{ij}^{of}(I_k^{-1}) \| x_{fjkl} \| \\ \| e_{fikl}(s, x_{kl}) \| &\leq \sum_{j=1}^{N} (\alpha_{ij}^{fo}(I_k^{-1}) \| x_{ojkl} \| + \alpha_{ij}^{ff}(I_k^{-1}) \| x_{fjkl} \|) \end{aligned} \tag{3.26}$$

where α_{ij}^{of}, α_{ij}^{fo} and α_{ij}^{ff} are polynomials in I_k^{-1} with the smallest power of I_k^{-1} denoted $\mathcal{O}(\cdot)$. $\mathcal{O}(\cdot)$ for these polynomials can be calculated from (3.23) as shown in Table 3.1.

To start analysis of the open-loop behavior of $\mathcal{S}_i$, we first write the solution of (3.21) as

$$\begin{aligned} x_{oikl}(s) &= e^{A_{oi}\tau_k s} x_{iokl}(0) + \xi_{iokl}(s) \\ x_{fikl}(s) &= e^{A_{fik} s} x_{fikl}(s) + \xi_{fikl}(s) + \tau_k b_{fik}(s) w_{ikl} \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} \xi_{iokl}(s) &= \int_0^s e^{A_{oi}\tau_k (s-z)} e_{oikl}\big(z, x_{fkl}(z)\big) dz \\ \xi_{fikl}(s) &= \int_0^s e^{A_{fik}(s-z)} e_{fikl}\big(z, x_{kl}(z)\big) dz \end{aligned} \tag{3.28}$$

and

$$b_{fik}(s) = \tau_k \int_0^s e^{A_{fik} z} b_{fi} dz \tag{3.29}$$

We now try to obtain bounds on $\| \xi_{oikl} \|$ and $\| \xi_{fikl} \|$ in (3.28). For this purpose, we first rewrite (3.21) in compact form as

$$\dot{x}_{kl}(s) = E\big(s, x_{kl}(s), w_{kl}\big) \tag{3.30}$$

where

$$
\begin{aligned}
x_{kl} &= \operatorname{col}[x_{oikl}, x_{fikl}] \\
w_{kl} &= \operatorname{col}[w_{ikl}]
\end{aligned}
$$

and $E(s, x_{kl}, v_{kl})$ is defined accordingly. Then

$$
x_{kl}(s) = x_{kl}(0) + \int_0^s E\big(z, x_{kl}(z), w_{kl}\big)dz \tag{3.31}
$$

Taking the norm of both sides of (3.31), and noting that $\| x_{fkl}(s) \|$ dominates norms of other terms involving $\| x_{kl}(s) \|$, we obtain

$$
\| x_{kl}(s) \| \leq \| x_{kl}(0) \| + \int_0^s (\alpha_x \| x_{kl}(z) \| + \tau_k \alpha_w \| w_{kl} \|)dz \tag{3.32}
$$

We use a variation of Gronwall Lemma [4] to convert (3.32) to an explicit inequality in $\| x_{kl}(s) \|$. For this purpose, we define

$$
\eta(s) = \| x_{kl}(0) \| + \int_0^s \big(\alpha_x \| x_{kl}(z) \| + \tau_k \alpha_w \| w_{kl} \| \big)dz
$$

and

$$
\kappa(s) = e^{-\alpha_x s}\eta(s) - \int_0^s \tau_k \alpha_w e^{-\alpha_x z} \| w_{kl} \| \, dz
$$

Then

$$
\kappa(0) = \eta(0) = \| x_{kl}(0) \|
$$

and

$$
\dot{\kappa}(s) = \alpha_x e^{-\alpha_x s}[\| x_{kl}(s) \| - \eta(s)] \leq 0
$$

so that

$$
\kappa(s) \leq \| x_{kl}(0) \|
$$

which implies

$$
\begin{aligned}
\| x_{kl}(s) \| &\leq \eta(s) \leq e^{\alpha_x s}\big(\| x_{kl}(0) \| + \int_0^s \tau_k \alpha_w e^{-\alpha_x z} \| w_{kl} \| \, dz\big) \\
&\leq \alpha^x \| x_{kl}(0) \| + \tau_k \alpha^w \| w_{kl} \|
\end{aligned} \tag{3.33}
$$

	$\mathcal{O}(\beta_{ij}^{oo})$	$\mathcal{O}(\beta_{ij}^{of})$	$\mathcal{O}(\beta_{ij}^{ow})$	$\mathcal{O}(\beta_{ij}^{fo})$	$\mathcal{O}(\beta_{ij}^{ff})$	$\mathcal{O}(\beta_{ij}^{fw})$
$m_i = 1,\, m_j = 1$	$2\nu_{max}$	ν_{max}	$2\nu_{max}$	ν_{max}	ν_{max}	$2\nu_{max}$
$m_i = 1,\, m_j \neq 1$	$2\nu_{max}$	$\nu_{max} + \nu$	$2\nu_{max} + \nu$	ν_{max}	ν_{max}	$2\nu_{max}$
$m_i \neq 1,\, m_j = 1$	$2\nu_{max}$	ν_{max}	$2\nu_{max}$	$\nu_{max} - \nu$	$\nu_{max} - \nu$	$2\nu_{max} - \nu$
$m_i \neq 1,\, m_j \neq 1$	$2\nu_{max}$	$\nu_{max} + \nu$	$2\nu_{max} + \nu$	ν_{max}	ν_{max}	$2\nu_{max}$

Table 3.2: Orders of β_{ij}^{oo}, β_{ij}^{of}, β_{ij}^{ov}, β_{ij}^{fo}, β_{ij}^{ff} and β_{ij}^{fv}

for some $\alpha^x > 0$ and $\alpha^w > 0$.

Now, the norm of $\xi_{fikl}(s)$ in (3.28) can be bounded as

$$\| \xi_{fikl}(s) \| \leq \sum_{j=1}^{N} (\beta_{ij}^{fo} \| x_{ojkl}(0) \| + \beta_{ij}^{ff} \| x_{fjkl}(0) \| + \beta_{ij}^{fw} \mid w_{jkl} \mid) \tag{3.34}$$

where the orders of the polynomials are found from (3.26) and (3.33) as

$$\begin{aligned} \mathcal{O}(\beta_{ij}^{fo}) &= \mathcal{O}(\beta_{ij}^{ff}) = \min\{\mathcal{O}(\alpha_{ij}^{fo}), \mathcal{O}(\alpha_{ij}^{ff})\} \\ \mathcal{O}(\beta_{ij}^{fw}) &= \nu_{max} + \min\{\mathcal{O}(\alpha_{ij}^{fo}), \mathcal{O}(\alpha_{ij}^{ff})\} \end{aligned} \tag{3.35}$$

These orders are tabulated in the second half of Table 3.2.

Although similar bounds can be obtained for $\| \xi_{oikl}(s) \|$, we can do better by first obtaining less conservative bounds on $x_{fikl}(s)$ than those given by (3.33), and then using these bounds in (3.28). From (3.27) and (3.34) we observe that

$$\| x_{fikl}(s) \| \leq \sum_{j=1}^{N} (\gamma_{ij}^{fo} \| x_{ojkl}(0) \| + \gamma_{ij}^{ff} \| x_{fjkl}(0) \| + \gamma_{ij}^{fw} \mid w_{jkl} \mid) \tag{3.36}$$

where γ_{ij}^{fo}, γ_{ij}^{ff} and γ_{ij}^{fw} are of the same order as β_{ij}^{fo}, β_{ij}^{ff} and β_{ij}^{fw} except that $\mathcal{O}(\gamma_{ii}^{ff}) = 0$ and $\mathcal{O}(\gamma_{ii}^{fw}) = \nu_{\max}$. Now, taking the norm of $\xi_{oikl}(s)$ in (3.28) and using (3.26) and (3.36), we obtain

$$\begin{aligned} \| \xi_{oikl}(s) \| &\leq \sum_{r=1}^{N} \alpha_{ir}^{of} \left[\sum_{j=1}^{N} \left(\gamma_{rj}^{fo} \| x_{ojkl}(0) \| + \gamma_{rj}^{ff} \| x_{fjkl}(0) \| + \gamma_{rj}^{fw} \mid w_{jkl} \mid \right) \right] \\ &\leq \sum_{j=1}^{N} \left(\beta_{ij}^{oo} \| x_{ojkl}(0) \| + \beta_{ij}^{of} \| x_{fjkl}(0) \| + \beta_{ij}^{ow} \mid w_{jkl} \mid \right) \end{aligned} \tag{3.37}$$

where

$$
\begin{aligned}
\beta_{ij}^{oo} &= \sum_{r=1}^{N} \alpha_{ir}^{of} \gamma_{rj}^{fo} \\
\beta_{ij}^{of} &= \sum_{r=1}^{N} \alpha_{ir}^{of} \gamma_{rj}^{ff} \\
\beta_{ij}^{ow} &= \sum_{r=1}^{N} \alpha_{ir}^{of} \gamma_{rj}^{fw}
\end{aligned} \tag{3.38}
$$

Using Table 3.1, second half of Table 3.2 (adapted for γ_{ij}^{ff} and γ_{ij}^{fw}) and (3.38), and considering all possibilities, we find out that

$$
\begin{aligned}
\mathcal{O}(\beta_{ij}^{oo}) &= 2\nu_{max} \\
\mathcal{O}(\beta_{ij}^{of}) &= \mathcal{O}(\alpha_{ij}^{of}) \\
\mathcal{O}(\beta_{ij}^{ow}) &= \nu_{max} + \mathcal{O}(\alpha_{ij}^{of})
\end{aligned} \tag{3.39}
$$

which are tabulated in the first half of Table 3.2.

Finally, for future use, we note from (3.27) that

$$
\| x_{oikl}(s) \| \leq \sum_{j=1}^{n} \left(\gamma_{ij}^{oo} \| x_{ojkl}(0) \| + \gamma_{ij}^{of} \| x_{fjkl}(0) \| + \gamma_{ij}^{ow} \mid w_{jkl} \mid \right) \tag{3.40}
$$

where γ_{ij}^{oo}, γ_{ij}^{of} and γ_{ij}^{ow} have the same orders as β_{ij}^{oo}, β_{ij}^{of} and β_{ij}^{ow} except that $\mathcal{O}(\gamma_{ii}^{oo}) = 0$.

Example 3.1.

Consider an interconnected system of $N = 6$ subsystems with $m_1 = m_2 = 1$, $m_3 = m_4 = 2$ and $m_5 = m_6 = 3$. Then

$$
\nu = 2, \quad \nu_1 = \nu_2 = 5, \quad \nu_3 = \nu_4 = 2, \quad \nu_5 = \nu_6 = 1
$$

Hence,

$$
T_{1k} = T_{2k} = \tau_k = \frac{1}{I_k^5}, \quad T_{3k} = T_{4k} = \frac{1}{I_k^2}, \quad T_{5k} = T_{6k} = T_k = \frac{1}{I_k}
$$

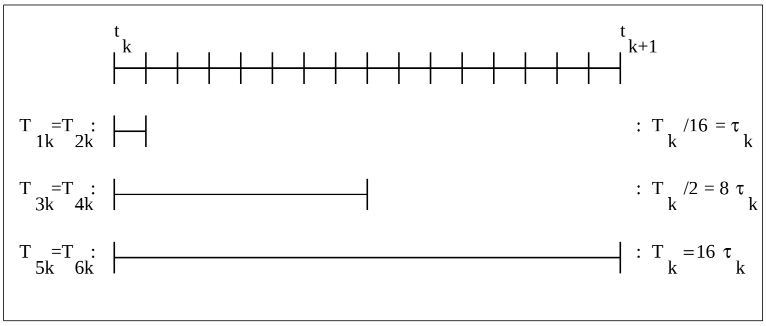

Figure 3.1: Relative lengths of T_{ik}, $i = 1, \dots, 6$

To illustrate relative lengths of T_{ik}, suppose $I_k = 2$. Then

$$T_{1k} = T_{2k} = \tau_k = \frac{1}{32}, \quad T_{3k} = T_{4k} = \frac{1}{4}, \quad T_{5k} = T_{6k} = T_k = \frac{1}{2}$$

Thus

$$M_{1k} = M_{2k} = 16, \quad M_{3k} = M_{4k} = 2, \quad M_{5k} = M_{6k} = 1$$

and

$$N_{1k} = N_{2k} = 1, \quad N_{3k} = N_{4k} = 8, \quad N_{5k} = N_{6k} = 16$$

Note that $N_{ik}M_{ik} = 16 = I_k^{\nu_{\max}-\nu_{\min}}$. Relative lengths of T_{ik} are shown in Figure 3.1.

Orders of $(\alpha_{ij}^{of}, \alpha_{ij}^{fo}, \alpha_{ij}^{ff})$, $(\beta_{ij}^{oo}, \beta_{ij}^{of}, \beta_{ij}^{ow})$, $(\beta_{ij}^{fo}, \beta_{ij}^{ff}, \beta_{ij}^{fw})$, $(\gamma_{ij}^{oo}, \gamma_{ij}^{of}, \gamma_{ij}^{ow})$ and $(\gamma_{ij}^{fo}, \gamma_{ij}^{ff}, \gamma_{ij}^{fw})$ are calculated from Table 3.1 and Table 3.2, are tabulated in Table 3.3-3.7.

(3.27) describes the continuous-time behavior of the open-loop interconnected system over a basic unit interval $t_k + l\tau_k \le t \le t_k + (l+1)\tau_k$. To describe the behavior of the subsystems at the discrete instants $t_k + l\tau_k$, we let $l = pN_{ik} + q$, $p = 0, 1, \dots, M_{ik} - 1$, $q = 0, 1, \dots, N_{ik} - 1$ and define the discrete-time states

$$\begin{aligned} x_{oi}[k,p,q] &= x_{oik,pN_{ik}+q}(0) &&= x_{oi}(t_k + pT_{ik} + q\tau_k) \\ x_{fi}[k,p,q] &= x_{fik,pN_{ik}+q}(0) &&= D_{fik}^{-1} x_{fi}(t_k + pT_{ik} + q\tau_k) \end{aligned} \tag{3.41}$$

j / i	1,2	3-6
1,2	(5,5,5)	(7,5,7)
3-6	(5,5,3)	(7,5,5)

Table 3.3: Orders of α_{ij}^{of}, α_{ij}^{fo}, and α_{ij}^{ff}

j / i	1,2	3-6
1,2	(10,5,10)	(10,7,12)
3-6	(10,5,10)	(10,7,12)

Table 3.4: Orders of β_{ij}^{oo}, β_{ij}^{of}, and β_{ij}^{ow}

j / i	1,2	3-6
1,2	(5,5,10)	(5,5,10)
3-6	(3,3,8)	(5,5,10)

Table 3.5: Orders of β_{ij}^{fo}, β_{ij}^{ff}, and β_{ij}^{fw}

	1	2	3	4	5	6
1	(0,5,10)	(10,5,10)	(10,7,12)	(10,7,12)	(10,7,12)	(10,7,12)
2	(10,5,10)	(0,5,10)	(10,7,12)	(10,7,12)	(10,7,12)	(10,7,12)
3	(10,5,10)	(10,5,10)	(0,7,12)	(10,7,12)	(10,7,12)	(10,7,12)
4	(10,5,10)	(10,5,10)	(10,7,12)	(0,7,12)	(10,7,12)	(10,7,12)
5	(10,5,10)	(10,5,10)	(10,7,12)	(10,7,12)	(0,7,12)	(10,7,12)
6	(10,5,10)	(10,5,10)	(10,7,12)	(10,7,12)	(10,7,12)	(0,7,12)

Table 3.6: Orders of γ_{ij}^{oo}, γ_{ij}^{of}, and γ_{ij}^{ow}

	1	2	3	4	5	6
1	(5,0,5)	(5,5,10)	(5,5,10)	(5,5,10)	(5,5,10)	(5,5,10)
2	(5,5,10)	(5,0,5)	(5,5,10)	(5,5,10)	(5,5,10)	(5,5,10)
3	(3,3,8)	(3,3,8)	(5,0,5)	(5,5,10)	(5,5,10)	(5,5,10)
4	(3,3,8)	(3,3,8)	(5,5,10)	(5,0,5)	(5,5,10)	(5,5,10)
5	(3,3,8)	(3,3,8)	(5,5,10)	(5,5,10)	(5,0,5)	(5,5,10)
6	(3,3,8)	(3,3,8)	(5,5,10)	(5,5,10)	(5,5,10)	(5,0,5)

Table 3.7: Orders of γ_{ij}^{fo}, γ_{ij}^{ff}, and γ_{ij}^{fw}

Note that for $p = 0, 1, \ldots, M_{ik} - 1$

$$\begin{aligned} x_{oi}[k, p, N_{ik}] &= x_{oi}[k, p+1, 0] \\ x_{fi}[k, p, N_{ik}] &= x_{fi}[k, p+1, 0] \end{aligned} \tag{3.42}$$

and for $p = M_{ik}$

$$\begin{aligned} x_{oi}[k, M_{ik}, N_{ik}] &= x_{oi}[k+1, 0, 0] \\ x_{fi}[k, M_{ik}, N_{ik}] &= D_{fik}^{-1} D_{fi,k+1} x_{fi}[k+1, 0, 0] \end{aligned} \tag{3.43}$$

Evolution of $x_{oi}[k, p, q]$ and $x_{fi}[k, p, q]$ can be found by evaluating (3.27) at $s = 1$, which gives

$$\begin{aligned} x_{oi}[k, p, q+1] &= e^{A_{oi}\tau_k} x_{oi}[k, p, q] + \xi_{oi}[k, p, q] \\ x_{fi}[k, p, q+1] &= e^{A_{fik}} x_{fi}[k, p, q] + \xi_{fi}[k, p, q] + \tau_k \Gamma_{fik} w_{ik, pN_{ik}+q} \end{aligned} \tag{3.44}$$

where $\xi_{oi}[k, p, q+1]$ and $\xi_{fi}[k, p, q+1]$ are obtained from (3.28) with $l = pN_{ik} + q$ and $s = 1$ and Γ_{fik} from (3.29) as

$$\Gamma_{fik} = \int_0^1 e^{A_{fik} z} b_{fi} dz \tag{3.45}$$

Note that, from (3.34) and (3.37), we have

$$\begin{aligned} \| \xi_{oi}[k, p, q+1] \| &\leq \sum_{j=1}^{N} \left(\beta_{ij}^{oo} \| x_{ojkl}(0) \| + \beta_{ij}^{of} \| x_{fjkl}(0) \| + \beta_{ij}^{ow} \mid w_{jkl} \mid \right) \\ \| \xi_{fi}[k, p, q+1] \| &\leq \sum_{j=1}^{N} \left(\beta_{ij}^{fo} \| x_{ojkl}(0) \| + \beta_{ij}^{ff} \| x_{fjkl}(0) \| + \beta_{ij}^{fw} \mid w_{jkl} \mid \right) \end{aligned} \tag{3.46}$$

for $q = 0, 1, \ldots, N_{ik} - 1$, where $l = pN_{ik} + q$.

For fixed k and p, solution of (3.44) for $q = 0, 1, \ldots, N_{ik} - 1$ is obtained as

$$\begin{aligned} x_{oi}[k,p,q] &= e^{A_{oi}q\tau_k} x_{oi}[k,p,0] + \sum_{r=0}^{q-1} e^{A_{oi}(q-1-r)\tau_k} \xi_{oi}[k,p,r] \\ x_{fi}[k,p,q] &= e^{A_{fik}q} x_{fi}[k,p,0] + \sum_{r=0}^{q-1} e^{A_{fik}(q-1-r)\tau_k} \xi_{oi}[k,p,r] \\ &+ \sum_{r=0}^{q-1} \tau_k e^{A_{fik}(q-1-r)} \Gamma_{fik} v_{ik,pN_{ik}+1} \end{aligned} \tag{3.47}$$

Evaluating (3.47) for $q = N_{ik}$, noting that

$$\begin{aligned} N_{ik}\tau_k &= T_{ik} \\ A_{fik}N_{ik} &= A_{fi} \end{aligned}$$

and

$$w_{ik,pN_{ik}+r} = w_{ik,pN_{ik}}, \quad r = 0, 1, \ldots, N_{ik} - 1$$

we obtain

$$\begin{aligned} x_{oi}[k,p,N_{ik}] &= e^{A_{oi}T_{ik}} x_{oi}[k,p,0] + \sum_{r=0}^{N_{ik}-1} e^{A_{oi}(N_{ik}-1-r)} \xi_{oi}[k,p,r] \\ x_{fi}[k,p,N_{ik}] &= e^{A_{fi}} x_{fi}[k,p,0] + \sum_{r=0}^{N_{ik}-1} e^{A_{fik}(N_{ik}-1-r)} \xi_{fi}[k,p,r] \\ &+ \sum_{r=0}^{N_{ik}-1} \tau_k e^{A_{fik}(N_{ik}-1-r)} \Gamma_{fik} w_{ik,pN_{ik}} \end{aligned} \tag{3.48}$$

Defining

$$\Gamma_{fi} = \int_0^1 e^{A_{fi}z} b_{fi} dz$$

and noting that

$$
\begin{aligned}
T_{ik}\Gamma_{fi} &= \sum_{r=0}^{N_{ik}-1} T_{ik} \int_{\frac{r}{N_{ik}}}^{\frac{r+1}{N_{ik}}} e^{A_{fik}N_{ik}z} b_{fi} dz \\
&= \sum_{r=0}^{N_{ik}-1} \frac{T_{ik}}{N_{ik}} \int_0^1 e^{A_{fik}(s+r)} b_{fi} ds \\
&= \sum_{r=0}^{N_{ik}-1} \tau_k e^{A_{fik}r} \int_0^1 e^{A_{fik}z} b_{fi} dz \\
&= \sum_{r=0}^{N_{ik}-1} \tau_k e^{A_{fik}(N_{ik}-1-r)} \Gamma_{fik} \qquad (3.49)
\end{aligned}
$$

(3.48) can be written as

$$
\begin{aligned}
\mathcal{S}_i^d : \quad x_{oi}[k,p+1] &= e^{A_{oi}T_{ik}} x_{oi}[k,p] + \xi_{oi}[k,p] \\
x_{fi}[k,p+1] &= e^{A_{fi}} x_{fi}[k,p] + \xi_{fi}[k,p] + T_{ik}\Gamma_{fi} w_i[k,p] \qquad (3.50)
\end{aligned}
$$

where

$$
\begin{aligned}
x_{oi}[k,p] &= x_{oi}[k,p,0] \\
x_{fi}[k,p] &= x_{fi}[k,p,0] \\
w_i[k,p] &= w_{ik,pN_{ik}} \qquad (3.51)
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_{oi}[k,p] &= \sum_{r=0}^{N_{ik}-1} e^{A_{oi}(N_{ik}-1-r)} \xi_{oi}[k,p,r] \\
\xi_{fi}[k,p] &= \sum_{r=0}^{N_{ik}-1} e^{A_{fik}(N_{ik}-1-r)} \xi_{fi}[k,p,r] \qquad (3.52)
\end{aligned}
$$

(3.50) constitutes the discrete model of $\mathcal{S}_i$ at local sampling instants. To complete the model, we need to obtain bounds on the $\xi_{oi}[k,p]$ and $\xi_{fi}[k,p]$ terms which represent the discrete-time effects of interconnections. However, since they depend not only on x_o and x_f but also on w_{kl}, we postpone this to the next section until after we obtain a model for the closed-loop system.

3.3 Decentralized Controllers and The Closed-Loop System

We generate local control inputs $w_i[k,p]$ in (3.50) by the discrete version of the decentralized controllers in (2.52) which are described as

$$\begin{aligned} \mathcal{C}_i^d: \quad x_{ci}[k,p+1] &= A_{ci}x_{ci}[k,p] + T_{ik}^{1-m_i} b_{ci} y_i(t_k + pT_{ik}) \\ w_i[k,p] &= T_{ik}^{-1} c_{ci}^T x_{ci}[k,p] + T_{ik}^{-m_i} d_{ci} y_i(t_k + pT_{ik}) \end{aligned} \tag{3.53}$$

where $x_{ci}[k,p] \in \Re^{m_i-1}$ is the state of $\mathcal{C}_i^d$ at the local sampling instant $t_k + pT_{ik}$ with the convention that

$$x_{ci}[k, M_{ik}] = x_{ci}[k+1, 0].$$

Using

$$\begin{aligned} y_i(t_k + pT_{ik}) &= c_{if}^T x_{fi}(t_k + pT_{ik}) = c_{if}^T D_{fik} x_{fi}[k,p] \\ &= T_{ik}^{m_i-1} c_{if}^T x_{fi}[k,p] \end{aligned}$$

the closed-loop subsystem $\hat{\mathcal{S}}_i^d$ consisting of $\mathcal{S}_i^d$ in (3.50) and $\mathcal{C}_i^d$ in (3.53) is described as

$$\begin{aligned} \hat{\mathcal{S}}_i^d: \quad \hat{x}_{oi}[k,p+1] &= \hat{\Phi}_{oi}\hat{x}_{oi}[k,p] + \hat{\xi}_{oi}[k,p] \\ \hat{x}_{fi}[k,p+1] &= \hat{\Phi}_{fi}\hat{x}_{fi}[k,p] + \hat{\xi}_{fi}[k,p] \end{aligned} \tag{3.54}$$

where

$$\begin{aligned} \hat{x}_{oi}[k,p] &= x_{oi}[k,p], \quad \hat{\xi}_{oi}[k,p] = \xi_{oi}[k,p] \\ \hat{x}_{fi}[k,p] &= \begin{bmatrix} x_{fi}[k,p] \\ x_{ci}[k,p] \end{bmatrix}, \quad \hat{\xi}_{fi}[k,p] = \begin{bmatrix} \xi_{fi}[k,p] \\ 0 \end{bmatrix} \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} \hat{\Phi}_{oi} &= e^{A_{oi}T_{ik}} \\ \hat{\Phi}_{fi} &= \begin{bmatrix} e^{A_{fi}} + \Gamma_{fi} d_{ci} c_{fi}^T & \Gamma_{fi} c_{ci}^T \\ b_{ci} c_{fi}^T & A_{ci} \end{bmatrix} \end{aligned}$$

Solution of (3.54) is given by

$$\begin{aligned}\hat{x}_{oi}[k,p] &= \hat{\Phi}_{oi}^{p}\hat{x}_{oi}[k,0]+\sum_{s=0}^{p-1}\hat{\Phi}_{oi}^{p-1-s}\hat{\xi}_{oi}[k,s]\\ \hat{x}_{fi}[k,p] &= \hat{\Phi}_{fi}^{p}\hat{x}_{fi}[k,0]+\sum_{s=0}^{p-1}\hat{\Phi}_{fi}^{p-1-s}\hat{\xi}_{fi}[k,s]\end{aligned} \tag{3.56}$$

Evaluating (3.56) for $p = M_{ik}$ and noting that

$$\begin{aligned}\hat{x}_{oi}[k,M_{ik}] &= \hat{x}_{oi}[k+1,0]\\ \hat{x}_{fi}[k,M_{ik}] &= D_{ik}^{-1}D_{i,k+1}\hat{x}_{fi}[k+1,0]\end{aligned} \tag{3.57}$$

where

$$D_{ik}=\begin{bmatrix} D_{fik} & \\ & I \end{bmatrix}$$

the behavior of $\hat{\mathcal{S}}_i^d$ over a common sampling interval is described by the discrete-time model

$$\begin{aligned}\hat{\mathcal{S}}_i^d:\quad \hat{x}_{oi}[k+1] &= \hat{\Phi}_{oi}^{M_{ik}}\hat{x}_{oi}[k]+\hat{\xi}_{oi}[k]\\ \hat{x}_{fi}[k+1] &= \hat{\Phi}_{fi}^{M_{ik}}\hat{x}_{fi}[k]+\hat{\xi}_{fi}[k]\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}\hat{x}_{oi}[k] &= \hat{x}_{oi}[k,0]\\ \hat{x}_{fi}[k] &= \hat{x}_{fi}[k,0]\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\hat{\xi}_{oi}[k] &= \sum_{s=0}^{M_{ik}-1}\hat{\Phi}_{oi}^{M_{ik}-1-s}\hat{\xi}_{oi}[k,s]\\ \hat{\xi}_{fi}[k] &= (D_{i,k+1}^{-1}D_{ik}-I)\hat{\Phi}_{fi}^{M_{ik}}\hat{x}_{fi}[k]\\ &+ D_{i,k+1}^{-1}D_{ik}\sum_{s=0}^{M_{ik}-1}\hat{\Phi}_{fi}^{M_{ik}-1-s}\hat{\xi}_{fi}[k,s]\end{aligned} \tag{3.60}$$

Note that

$$\hat{\Phi}_{oi}^{M_{ik}} = e^{A_{oi}M_{ik}T_{ik}} = e^{A_{ik}T_k},\quad i=1,2,\ldots,N \tag{3.61}$$

To complete the closed-loop discrete-time model in (3.58), we need to obtain suitable bounds on the interconnection terms $\hat{\xi}_{oi}[k]$ and $\hat{\xi}_{fi}[k]$ in (3.58) in terms of $\hat{x}_{oi}[k]$ and $\hat{x}_{fi}[k]$. For this purpose, we first obtain bounds of $\xi_{oi}[k,p,q]$ and $\xi_{fi}[k,p,q]$ in (3.44) for a fixed p and for $q = 0, 1, \ldots, N_{ik} - 1$, then use (3.52) and (3.55) to obtain bounds for $\hat{\xi}_{oi}[k,s]$ and $\hat{\xi}_{fi}[k,s]$ in (3.56) for $s = 0, 1, \ldots, M_{ik} - 1$ and finally (3.60) to obtain bounds of $\hat{\xi}_{oi}[k]$ and $\hat{\xi}_{fi}[k]$. The crucial point is to eliminate all the intermediate variables $\mid x_{oikl}(0) \mid$, $\mid x_{fikl}(0) \mid$ and $\mid w_{ikl} \mid$ that appear in the expressions for $\xi_{oi}[k,p,q]$ and $\xi_{fi}[k,p,q]$. $\mid w_{ikl} \mid$ can easily replaced with appropriate bounds on $\mid \hat{x}_{fi}[k,p,q] \mid$ by using (3.47) and (3.53), that is

$$\mid w_{ikl} \mid \leq \mathcal{O}(T_{ik}^{-1}) \parallel \hat{x}_{fi}[k,p] \parallel, \quad pN_{ik} \leq l < (p+1)N_{ik} \tag{3.62}$$

However, elimination of $\mid x_{oikl}(0) \mid$ and $\mid x_{fikl}(0) \mid$ requires that we should keep track of them by using (3.36) and (3.40). We illustrate the elimination procedure for the typical case considered in Example 3.1, where the subsystems are ordered in increasing T_{ik} (decreasing ν_i), which is important in elimination of the intermediate variable in a systematic way.

We start with $l = 1$, which corresponds to $p = 0$, $q = 1$ for all the subsystems and for which we have

$$\begin{aligned}
\parallel \xi_{oi}[k,0,1] \parallel &\leq \sum_{j=1}^{N} \left(\beta_{ij}^{oo} \parallel x_{ojk0}(0) \parallel + \beta_{ij}^{of} \parallel x_{fjk0}(0) \parallel + \beta_{ij}^{ow} \mid w_{jk0} \mid \right) \\
\parallel \xi_{fi}[k,0,1] \parallel &\leq \sum_{j=1}^{N} \left(\beta_{ij}^{fo} \parallel x_{ojk0}(0) \parallel + \beta_{ij}^{ff} \parallel x_{fjk0}(0) \parallel + \beta_{ij}^{fw} \mid w_{jk0} \mid \right)
\end{aligned} \tag{3.63}$$

Substituting

$$\begin{aligned}
\parallel x_{ojk0}(0) \parallel &= \parallel \hat{x}_{oj}[k,0] \parallel \\
\parallel x_{fjk0}(0) \parallel &\leq \parallel \hat{x}_{fj}[k,0] \parallel \\
\mid w_{jk0} \mid &\leq \mathcal{O}(T_{jk}^{-1}) \parallel \hat{x}_{fj}[k,0] \parallel
\end{aligned}$$

and noting that

$$\begin{aligned}
\min\{\mathcal{O}(\beta_{ij}^{of}), \mathcal{O}(\beta_{ij}^{ow}) + \mathcal{O}(T_{jk}^{-1})\} &= \mathcal{O}(\beta_{ij}^{of}) \\
\min\{\mathcal{O}(\beta_{ij}^{ff}), \mathcal{O}(\beta_{ij}^{fw}) + \mathcal{O}(T_{jk}^{-1})\} &= \mathcal{O}(\beta_{ij}^{ff})
\end{aligned} \tag{3.64}$$

We rewrite (3.63) as

$$\begin{aligned}
\| \xi_{oi}[k,0,1] \| &\leq \sum_{j=1}^{N} \beta_{ij}^{oo} \| \hat{x}_{oj}[k,0] \| + \beta_{ij}^{of} \| \hat{x}_{fj}[k,0] \| \\
\| \xi_{fi}[k,0,1] \| &\leq \sum_{j=1}^{N} \beta_{ij}^{fo} \| \hat{x}_{oj}[k,0] \| + \beta_{ij}^{ff} \| \hat{x}_{fj}[k,0] \|
\end{aligned} \tag{3.65}$$

Note that β's in (3.63) and (3.65) are not the same. However, they are of the same order and we used the same symbol not to introduce more complexity in the notation.

We also need bounds of $\| x_{oik1}(0) \|$ and $\| x_{fik1}(0) \|$ to be used in the next step. Using (3.36) and (3.40) and noting that (3.64) is also valid for γ's, we similarly obtain

$$\begin{aligned}
\| x_{oik1}(0) \| &\leq \sum_{j=1}^{N} \gamma_{ij}^{oo} \| \hat{x}_{oj}[k,0] \| + \gamma_{ij}^{of} \| \hat{x}_{ij}[k,0] \| \\
\| x_{fik1}(0) \| &\leq \sum_{j=1}^{N} \gamma_{ij}^{fo} \| \hat{x}_{oj}[k,0] \| + \gamma_{ij}^{ff} \| \hat{x}_{ij}[k,0] \|
\end{aligned} \tag{3.66}$$

Before proceeding any further, we also note that for $i = 1, 2$ (for which $N_{ik} = 1$), (3.65) and (3.66) can also be interpreted as

$$\left.\begin{aligned}
\| \hat{\xi}_{oi}[k,1] \| &\leq \textstyle\sum_{j=1}^{N} \beta_{ij}^{oo} \| \hat{x}_{oj}[k,0] \| + \beta_{ij}^{of} \| \hat{x}_{fj}[k,0] \| \\
\| \hat{\xi}_{fi}[k,1] \| &\leq \textstyle\sum_{j=1}^{N} \beta_{ij}^{fo} \| \hat{x}_{oj}[k,0] \| + \beta_{ij}^{ff} \| \hat{x}_{fj}[k,0] \|
\end{aligned}\right\} i = 1, 2 \tag{3.67}$$

and

$$\left.\begin{aligned}
\| \hat{x}_{oi}[k,1] \| &\leq \textstyle\sum_{j=1}^{N} \gamma_{ij}^{oo} \| \hat{x}_{oj}[k,0] \| + \gamma_{ij}^{of} \| \hat{x}_{fj}[k,0] \| \\
\| \hat{x}_{fi}[k,1] \| &\leq \textstyle\sum_{j=1}^{N} \gamma_{ij}^{fo} \| \hat{x}_{oj}[k,0] \| + \gamma_{ij}^{ff} \| \hat{x}_{ij}[k,0] \|
\end{aligned}\right\} i = 1, 2 \tag{3.68}$$

Now, let $l = 2$, which corresponds to

$$p = 1, q = 1 \quad \text{for} \quad i = 1, 2$$

$$p = 0, q = 2 \quad \text{for} \quad i = 3 - 6$$

and therefore, requires separate analysis for $i, j = 1, 2$ and for $i, j = 3 - 6$. For $i = 1, 2$, we have

$$\begin{aligned}\hat{\xi}_{oi}[k,2] &\leq \sum_{j=1}^{2} \beta_{ij}^{oo} \parallel \hat{x}_{oj}[k,1] \parallel + \beta_{ij}^{of} \parallel \hat{x}_{fj}[k,1] \parallel + \beta_{ij}^{ow} \mid w_{jk1} \mid \\ &+ \sum_{j=3}^{N} \beta_{ij}^{oo} \parallel x_{ojk1}(1) \parallel + \beta_{ij}^{of} \parallel x_{fjk1}(1) \parallel + \beta_{ij}^{ow} \mid w_{jk0} \mid \end{aligned} \tag{3.69}$$

Using

$$\mid w_{jk1} \mid \leq \mathcal{O}(T_{jk}^{-1}) \parallel \hat{x}_{fj}[k,1] \parallel, \quad j = 1, 2$$

and (3.64), the last two terms in the first sum above can be combined under $\beta_{ij}^{of} \parallel \hat{x}_{fj}[k,1] \parallel$. Substituting $\parallel x_{ojk1}(1) \parallel$ and $\parallel x_{fjk1}(1) \parallel$ from (3.66), (3.69) becomes

$$\begin{aligned}\hat{\xi}_{oi}[k,2] &\leq \sum_{j=1}^{2} \beta_{ij}^{oo} \parallel \hat{x}_{oj}[k,1] \parallel + \beta_{ij}^{of} \parallel \hat{x}_{fj}[k,1] \parallel \\ &+ \sum_{r=1}^{N} \left(\sum_{j=3}^{N} \beta_{ij}^{oo}\gamma_{jr}^{oo} + \beta_{ij}^{of}\gamma_{jr}^{fo} \right) \parallel \hat{x}_{or}[k,0] \parallel \\ &+ \sum_{r=1}^{N} \left(\sum_{j=3}^{N} \beta_{ij}^{oo}\gamma_{jr}^{of} + \beta_{ij}^{of}\gamma_{jr}^{ff} \right) \parallel \hat{x}_{fr}[k,0] \parallel \\ &+ \sum_{j=3}^{N} \beta_{ij}^{ow} \mid w_{jk0} \mid \end{aligned} \tag{3.70}$$

Using tables 3.1-3.4, it can be shown that

$$\begin{aligned}\mathcal{O}\left(\sum_{j=3}^{N} \beta_{ij}^{oo}\gamma_{jr}^{oo} + \beta_{ij}^{of}\gamma_{jr}^{fo} \right) &= \mathcal{O}(\beta_{ir}^{oo}) \\ \mathcal{O}\left(\sum_{j=3}^{N} \beta_{ij}^{oo}\gamma_{jr}^{of} + \beta_{ij}^{of}\gamma_{jr}^{ff} \right) &= \begin{cases} \mathcal{O}(\beta_{ir}^{of}) + \nu_{max} & i = 1, 2 \\ \mathcal{O}(\beta_{ir}^{of}) & i = 3 - 6 \end{cases}\end{aligned} \tag{3.71}$$

Assimilating $\mid w_{jkl} \mid$ terms for $j = 3 - 6$ in $\parallel \hat{x}_{fj}[k,0] \parallel$ terms with the help of (3.64), substituting the expressions for $\parallel \hat{x}_{oj}[k,1] \parallel$ and $\parallel \hat{x}_{fj}[k,1] \parallel$ from (3.66), and using (3.71), (3.70) eventually reduces to

$$\hat{\xi}_{oi}[k,2] \leq \sum_{j=1}^{N} \left(\beta_{ij}^{oo} \parallel \hat{x}_{oj}[k,0] \parallel + \beta_{ij}^{of} \parallel \hat{x}_{fj}[k,0] \parallel \right), \quad i = 1, 2 \tag{3.72}$$

Similarly, we can bound $\hat{\xi}_{fi}[k,2]$, $i = 1,2$, by exactly the same expression with β_{ij}^{oo} and β_{ij}^{of} replaced with β_{ij}^{fo} and β_{ij}^{ff}. Clearly, the same expression is also valid for $i = 3-6$, except that the left-hand sides are $\xi_{oi}[k,0,2]$ and $\xi_{fi}[k,0,2]$. Finally, the bounds of $\| \hat{x}_{oi}[k,2] \|$ and $\| \hat{x}_{fi}[k,2] \|$ for $i = 1,2$; and of $\| x_{oik2}(0) \|$ and $\| x_{fik2}(0) \|$ are given by the same expressions with β's replaced with γ's.

The analysis above shows that the perturbation terms at any discrete instant $t = t_k + l\tau_k$ are bounded by β's times corresponding initial discrete states at $t = t_k$. Hence, $\hat{\xi}_{oi}[k]$ and $\hat{\xi}_{fi}[k]$ in (3.58) are bounded as

$$
\begin{aligned}
\| \hat{\xi}_{oi}[k] \| &\leq \sum_{j=1}^{N} \beta_{ij}^{oo} \| \hat{x}_{oj}[k] \| + \beta_{ij}^{of} \| \hat{x}_{fj}[k] \| \\
\| \hat{\xi}_{fi}[k] \| &\leq \mathcal{O}\Big(\| D_{i,k+1}^{-1} D_{ik} - I \| \Big) \| \hat{x}_{fi}[k] \| \\
&+ \mathcal{O}\Big(\| D_{i,k+1}^{-1} D_{ik} \| \Big) \sum_{j=1}^{N} \beta_{ij}^{oo} \| \hat{x}_{oj}[k] \| + \beta_{ij}^{of} \| \hat{x}_{ij}[k] \| \qquad (3.73)
\end{aligned}
$$

Note that provided

$$\left(\frac{I_{k+1}}{I_k} \right)^{\nu} \leq c \qquad (3.74)$$

for any fixed $c > 1$, we have

$$\| D_{i,k+1}^{-1} D_{ik} \| \leq c$$

$$\| D_{i,k+1}^{-1} D_{ik} - I \| \leq c - 1$$

in which case (3.73) becomes

$$
\begin{aligned}
\| \hat{\xi}_{oi}[k] \| &\leq \sum_{j=1}^{N} \beta_{ij}^{oo} \| \hat{x}_{oj}[k] \| + \beta_{ij}^{of} \| \hat{x}_{fj}[k] \| \\
\| \hat{\xi}_{fi}[k] \| &\leq \sum_{j=1}^{N} \beta_{ij}^{fo} \| \hat{x}_{oj}[k] \| + \beta_{ij}^{ff} \| \hat{x}_{fj}[k] \| \qquad (3.75)
\end{aligned}
$$

3.4 Stabilization By Decentralized Control

Since $(A_{fi}, b_{fi}, c_{fi}^T)$ are controllable and observable with A_{fi} having all their eigenvalues at the origin, $(e^{A_{fi}}, \Gamma_{fi}, c_{fi}^T)$ are also controllable and observable. Then, the local controller parameters $(A_{ci}, b_{ci}, c_{ci}^T, d_{ci})$ can be chosen such that $\hat{\Phi}_{fi}$ in (3.56) have desired eigenvalues [2]. Let $\mathcal{C}_i^d$ be chosen to have $\hat{\Phi}_{fi}$ Schur stable, that is, with all eigenvalues within the unit circle [1] . Then, there exist positive definite matrices $\hat{P}_{fi}$ such that

$$\hat{\Phi}_{fi}^T \hat{P}_{fi} \hat{\Phi}_{fi} - \hat{P}_{fi} = -I, \quad i = 1, 2, \ldots, N \tag{3.78}$$

from which we also obtain

$$(\hat{\Phi}_{fi}^{M_{ik}})^T \hat{P}_{fi} (\hat{\Phi}^{M_{ik}}) - \hat{P}_{fi} = -I - \hat{\Phi}_{fi}^T \hat{\Phi}_{fi} - \cdots - (\hat{\Phi}^{M_{ik}-1})^T (\hat{\Phi}^{M_{ik}-1}) \tag{3.79}$$

On the other hand, since A_{oi} is Hurwitz stable by assumption, there exist positive definite matrices $\hat{P}_{oi}$ such that

$$A_{oi}^T \hat{P}_{oi} - \hat{P}_{oi} A_{oi} = -I \tag{3.80}$$

Then

$$\begin{aligned} (\hat{\Phi}_{oi}^{M_{ik}})^T \hat{P}_{oi} (\hat{\Phi}_{oi}^{M_{ik}}) - \hat{P}_{oi} &= \int_o^{T_k} \frac{d}{dt} \left(e^{A_{oi}^T t} \hat{P}_{oi} e^{A_{oi} t} \right) dt \\ &= -\int_0^{T_k} e^{A_{oi}^T t} \hat{P}_{oi} e^{A_{oi} t} dt \end{aligned} \tag{3.81}$$

so that

$$\hat{x}_{oi}^T \left[(\hat{\Phi}_{oi}^{M_{ik}})^T \hat{P}_{oi} (\hat{\Phi}_{oi}^{M_{ik}}) - \hat{P}_{oi} \right] \hat{x}_{oi} \leq -c_{oi} T_k \parallel \hat{x}_{oi} \parallel^2 \tag{3.82}$$

for some $c_{oi} > 0$ independent of T_k.

We now choose

$$\upsilon[k] = \sum_{i=1}^{N} \left(\hat{x}_{oi}^T[k] \hat{P}_{oi} \hat{x}_{oi}[k] + \hat{x}_{fi}^T[k] \hat{P}_{fi} \hat{x}_{fi}[k] \right)$$

---

[1] Note that

$$c_{fi}^T (zI - e^{A_{fi}})^{-1} \Gamma_{fi} = H_{fi}(z) \tag{3.76}$$

is the zero-order hold discrete equivalent of

$$H_{fi}(s) = \frac{1}{s^{m_i}} \tag{3.77}$$

with normalized sampling period $T_i = 1$ and approximates the zero-order hold equivalent of $\mathcal{S}_i$ at high frequencies.

as a Lyapunov function for the closed-loop discrete-time interconnected system in (3.58). Calculating

$$\Delta v[k] = v[k+1] - v[k]$$

along the solutions of (3.58) and using (3.75), (3.79) and (3.82), $\Delta v[k]$ can be majorized as

$$\Delta v \leq -z^T[k]\Big(I - Q(I_k)\Big)z[k] \tag{3.83}$$

where

$$z[k] = \text{col}\left[(c_{oi}I_k^{-\nu_m})^{\frac{1}{2}} \parallel \hat{x}_{oi}[k] \parallel, \parallel \hat{x}_{fi}[k] \parallel\right] \tag{3.84}$$

and $Q[I_k]$ is a symmetric matrix of the form

$$Q[I_k] = \begin{bmatrix} Q_{oo}[I_k] & Q_{of}[I_k] \\ Q_{of}^T[I_k] & Q_{ff}[k] \end{bmatrix} \tag{3.85}$$

with $Q_{oo}[I_k] = \left[q_{ij}^{oo}[I_k]\right]$ having the elements

$$q_{ij}^{oo}[I_k] = \mathcal{O}(I_k^{-\nu_{\max}})$$

$Q_{of}[I_k] = \left[q_{ij}^{of}[I_k]\right]$ the elements

$$q_{ij}^{of}[I_k] = \begin{cases} \mathcal{O}(I_k^{\nu - \frac{\nu_{\max}}{2}}), & m_i = 1, m_j \neq 1 \text{ or } m_i \neq 1, m_j = 1 \\ \mathcal{O}(I_k^{\frac{-\nu_{\max}}{2}}), & \text{otherwise} \end{cases}$$

and $Q_{ff}[I_k] = \left[q_{ij}^{ff}[I_k]\right]$ the elements

$$q_{ij}^{ff}[I_k] = \begin{cases} \mathcal{O}(I_k^{\nu-\nu_{\max}}), & m_i = 1, m_j \neq 1 \text{ or } m_i \neq 1, m_j = 1 \\ \mathcal{O}(I_k^{-\nu_{\max}}), & \text{otherwise} \end{cases}$$

Note that if $m_i = 1$ for any of the subsystems, then $\nu_{\max} = 2\nu + 1$ so that $\nu - \nu_{\max}/2 = -1/2$. Since all powers of I_k in each of the expressions above are negative it follows that there exists sufficiently large $I_* > 1$ that depends on the bounds of the interconnections in (3.3) such that $I - Q[I_k]$ is positive definite for all $I_k \geq I_*$. This establishes the (Shur) stability of the discrete-model of the closed-loop interconnected system.

3.5 Adaptation of Sampling Intervals and Controller Gains

In the previous section, we established that if the bounds of the interconnections are known, then we can find $I_\star$ such that the discrete model of the closed-loop system is stable for all $I_k > I_\star$. If the bounds of the interconnections are not known, then $I_\star$ is not known a priori and I_k has to be adjusted by some means until it reaches the desired unknown value of $I_\star$. A simple way of achieving this is to adjust I_k using the following rule:

$$\begin{aligned} \rho_{k+1} &= \rho_k + \min\{1, S_k\} \\ S_k &= d_y \parallel y(t_k) \parallel + d_c \parallel x_c(t_k) \parallel \\ I_k &= \mathrm{int}\,(\rho_k) \end{aligned} \tag{3.86}$$

This rule guarantees that I_k is non-decreasing and also

$$I_{k+1} \leq I_k + 1$$

so that

$$\big(\frac{I_{k+1}}{I_k}\big)^\nu \leq \big(1 + \frac{1}{I_k}\big)^\nu \leq 2^\nu$$

for any $I_k \geq 1$ and therefore (3.74) is also satisfied. However, there are two problems associated with the choice in (3.86).

The first problem is that I_k might increase indefinitely. In this case, $T_k = 1/I_k^{\nu_{\min}}$ will decrease forever and it is possible that

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} (t_0 + \sum_{l=0}^{k-1} T_k) = t_\infty < \infty$$

Then, the discrete model in (3.58) will represent the closed-loop system only on a finite interval $[t_0, t_\infty)$ and we cannot deduce stability of the actual sampled-data system from stability of the discrete model. In fact, arbitrarily large I_k is practically impossible, as that means infinitely fast sampling and arbitrarily high gains.

The second problem associated with the choice in (3.86) is that the degree of stability of the discrete-model might get smaller in successive common sampling intervals, resulting in poorer and poorer convergence of $\hat{x}[k]$. To see this consider (3.83), which implies that

$$\begin{aligned} \Delta v[k] &\leq -\lambda_{\min}\big(I - Q[I_k]\big) \parallel z[k] \parallel^2 \\ &\leq -\lambda_m \parallel z[k] \parallel^2 \end{aligned} \tag{3.87}$$

where

$$\lambda_m = \lambda_{\min}\big(I - Q[I_\star]\big)$$

Since

$$\parallel z[k] \parallel^2 \geq c_o I_k^{-\nu_{\min}} \parallel \hat{x}[k] \parallel^2 \geq c_o I_k^{-\nu_{\min}} \lambda_M^{-1} v[k]$$

where

$$\lambda_M = \max_i \{\lambda_{\max}(\hat{P}_{oi}), \lambda_{\max}(\hat{P}_{fi})\}$$

(3.87) implies that

$$v[k+1] \leq \sigma_k^2 v[k] \tag{3.88}$$

where

$$\sigma_k^2 = 1 - \frac{c_o \lambda_m}{I_k^{\nu_{\min}} \lambda_M} < 1 \quad \text{for} \quad I_k \geq I_{\min} \tag{3.89}$$

Hence

$$\parallel \hat{x}[k] \parallel \leq M\Big(\prod_{l=0}^{k-1} \sigma_l\Big) \parallel \hat{x}[0] \parallel, \quad M > 0 \tag{3.90}$$

which is the best bound on $\parallel \hat{x}[k] \parallel$ that we can obtain from Lyapunov analysis. Since $\sigma_k \to 1$ as $I_k \to \infty$, we observe that uncontrolled increase in I_k should be avoided.

To avoid the problems mentioned above, we propose to keep I_k unchanged for a fixed duration of time Δt that contains an integral number of every possible common sampling interval $T_k = I_k^{-\nu_{\min}}$. A convenient choice is $\Delta t = 1$, which contains

$$\frac{\Delta t}{T_k} = I_k^{\nu_{\min}} = M_k$$

common sampling intervals. Thus I_k is updated (if necessary) only at the discrete instants t_0, $t_0 + 1$, $t_0 + 2$, etc.

To analyze the stability properties of the closed-loop system described at $t_0 + k$, $k = 1, 2, \ldots$, we define new discrete-time state variables as

$$\begin{aligned}\hat{X}_{oi}[k] &= \hat{x}_{oi}[M_0 + \cdots + M_{k-1}] \\ \hat{X}_{fi}[k] &= \hat{x}_{fi}[M_0 + \cdots + M_{k-1}]\end{aligned} \tag{3.91}$$

for $k = 1, 2, \ldots$. Then from (3.58), we obtain

$$\begin{aligned}\hat{X}_{oi}[k+1] &= \hat{\Psi}_{oi}\hat{X}_{oi}[k] + \hat{\Xi}_{oi}[k] \\ \hat{X}_{fi}[k+1] &= \hat{\Psi}_{fi}\hat{X}_{fi}[k] + \hat{\Xi}_{fi}[k]\end{aligned} \tag{3.92}$$

where

$$\begin{aligned}\hat{\Psi}_{oi} &= \hat{\Phi}_{oi}^{M_k M_{ik}} &= e^{A_{oi} M_k T_k} = e^{A_{oi}} \\ \hat{\Psi}_{fi} &= \hat{\Phi}_{fi}^{M_k M_{ik}} &= \hat{\Phi}_{fi}^{I_k^{\nu_i}}\end{aligned} \tag{3.93}$$

and

$$\begin{aligned}\hat{\Xi}_{oi}[k] &= \sum_{\rho=0}^{M_k - 1} \hat{\Phi}_{oi}^{(M_k - 1 - \rho)M_{ik}} \hat{\xi}_{oi}[M_0 + \cdots + M_{k-1} + \rho] \\ \hat{\Xi}_{fi}[k] &= \sum_{\rho=0}^{M_k - 1} \hat{\Phi}_{fi}^{(M_k - 1 - \rho)M_{ik}} \hat{\xi}_{fi}[M_0 + \cdots + M_{k-1} + \rho]\end{aligned} \tag{3.94}$$

An analysis similar to the one carried out for $\hat{\xi}_{oi}[k]$ and $\hat{\xi}_{fi}[k]$ in the previous section reveals that provided I_k's satisfy (3.74), we have

$$\begin{aligned}\| \hat{\Xi}_{oi}[k] \| &\leq \sum_j \beta_{ij}^{oo} \| \hat{X}_{oj}[k] \| + \beta_{ij}^{of} \| \hat{X}_{fj}[k] \| \\ \| \hat{\Xi}_{fi}[k] \| &\leq \sum_j \beta_{ij}^{fo} \| \hat{X}_{oj}[k] \| + \beta_{ij}^{ff} \| \hat{X}_{fj}[k] \|\end{aligned} \tag{3.95}$$

We now proceed with the stability analysis of Section 3.4. However, this time we choose $\hat{P}_{oi}$ directly to satisfy

$$\hat{\Psi}_{oi}^T \hat{P}_{oi} \hat{\Psi}_{oi} - \hat{P}_{oi} = -I \tag{3.96}$$

which is possible as $\hat{\Psi}_{oi}$ in (3.93) are Schur-stable (independent of T_k). Using

$$V[k] = \sum_i \big(\hat{X}_{oi}[k]\hat{P}_{oi}\hat{X}_{oi}[k] + \hat{X}_{fi}[k]\hat{P}_{fi}\hat{X}_{fi}[k]\big) \tag{3.97}$$

as a Lyapunov function for the closed-loop discrete-time system in (3.92), we find that

$$\Delta V[k] \leq -Z^T[k]\big(I - \hat{Q}[k]\big)Z[k] \tag{3.98}$$

where now

$$Z[k] = \mathrm{col}\left[\ \| \hat{X}_{oi}[k] \|, \| \hat{X}_{fi}[k] \|\ \right]$$

and the blocks of

$$\hat{Q}[k] = \begin{bmatrix} \hat{Q}_{oo}[k] & \hat{Q}_{of}[k] \\ \hat{Q}_{of}^T[k] & \hat{Q}_{ff}[k] \end{bmatrix}$$

have the elements

$$\begin{aligned} \hat{q}_{ij}^{oo}[k] &= \mathcal{O}[I_k^{-\nu_{\max}}] \\ \hat{q}_{ij}^{of}[k] &= \begin{cases} \mathcal{O}[I_k^{\nu-\nu_{\max}}] & , m_i = 1, m_j \neq 1 \\ \mathcal{O}[I_k^{-\nu_{\max}}] & , otherwise \end{cases} \\ \hat{q}_{ij}^{ff}[k] &= \begin{cases} \mathcal{O}[I_k^{\nu-\nu_{\max}}] & , m_i = 1, m_j \neq 1 \\ \mathcal{O}[I_k^{-\nu_{\max}}] & , otherwise \end{cases} \end{aligned} \tag{3.99}$$

Again, there exists I_* such that $I - \hat{Q}[k]$ is positive definite for all $I_k \geq I_*$. However, the difference from the previous case is that I_k does not appear in the expression (3.89) for the degree of exponential stability σ_k. In other words, there exists fixed $\sigma_* < 1$ such that

$$\| \hat{X}[k] \| \leq M\sigma_*^{(k-k_0)} \| X[k_0] \| \tag{3.100}$$

for all $I_k \geq I_*$. This is exactly what prevents I_k from growing indefinitely under the adaptation rule in (3.86) as we explain below.

Suppose that $I_k \geq I_*$ for some k_*. Then $\hat{\mathcal{S}}^d$ is exponentially stable with degree of stability σ_* so that S_k in (3.86) satisfies

$$S_k \leq \eta \| \hat{X}[k] \|^2$$

for some $\eta > 0$. Using (3.100) with $k_0 = k_*$, we have

$$S_k \le \eta M^2 \sigma_*^{2(k-k_*)} \parallel \hat{X}[k_*] \parallel^2$$

for all $k \ge k_*$ so that

$$\begin{aligned} \rho_k &\le \rho_{k_*} + \sum_{l=k_*}^{k-1} S_k \\ &\le \rho_{k_*} + \eta M^2 \parallel \hat{X}[k_*] \parallel^2 \frac{1 - \sigma_*^{2(k-k_*)}}{1 - \sigma_*^2} \end{aligned}$$

Then $\lim_{k\to\infty} \rho_k < \infty$ and therefore

$$\lim_{k\to\infty} I_k = I_\infty < \infty$$

This guarantees stability of the closed-loop sampled-data system.

Chapter 4

AN EXAMPLE: COUPLED INVERTED PENDULI

Consider the system consisting of three coupled inverted penduli shown in Figure 4.1 [14]. We assume that first two penduli form a subsystem, while the third one a second subsystem interconnected with the first one through a coupling spring.

The system is modeled by three non-linear second order differential equations as

$$\begin{aligned} S_1 : m_{11} l_{11}^2 \ddot{\theta}_{11} &= m_{11} l_{11} g \sin\theta_{11} - k_{11}\theta_{11} - k_{1c}(\theta_{11} - \theta_{12}) - b_{11}\dot{\theta}_{11} \\ &- b_{1c}(\dot{\theta}_{11} - \dot{\theta}_{12}) + u_1 \\ m_{12} l_{12}^2 \ddot{\theta}_{12} &= m_{12} l_{12} g \sin\theta_{12} - k_{12}\theta_{12} + k_{1c}(\theta_{11} - \theta_{12}) - b_{12}\dot{\theta}_{12} \\ &+ b_{1c}(\dot{\theta}_{11} - \dot{\theta}_{12}) - k_c(\tan\theta_{12} - \tan\theta_2) \end{aligned} \quad (4.1)$$

$$S_2 : m_2 l_2^2 \ddot{\theta}_2 = m_2 l_2 \sin\theta_2 - k_2\theta_2 - b_2\dot{\theta}_2 + k_c(\tan\theta_{12} - \tan\theta_2) + u_2 \quad (4.2)$$

where θ_{11}, θ_{12} and θ are angular displacements of the penduli from the vertical equilibria and u_1 and u_2 are the external torques (inputs) applied to the first and third penduli. The parameters in (4.1) and (4.2) summarized in Table 4.1.

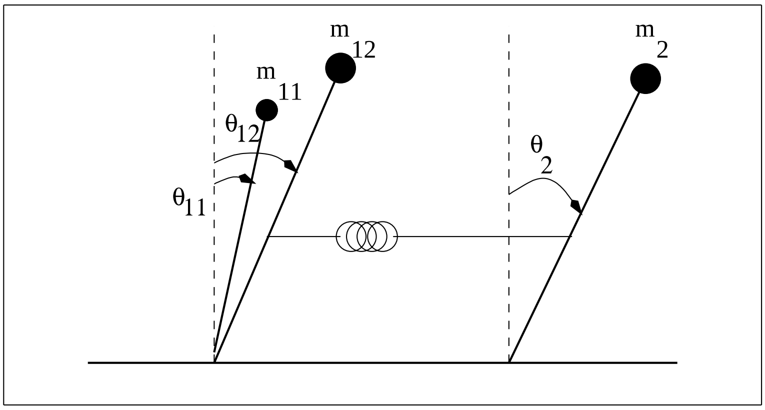

Figure 4.1: Three coupled inverted penduli

k_{11}, k_{12}, k_2	:spring coefficients
b_{11}, b_{12}, b_2	:damping coefficients
k_{1c}, b_{1c}	:spring and damping coefficients coupling m_{11} and m_{12}
k_c	:spring coefficient coupling m_{12} and m_2

Table 4.1: Parameters appearing in (4.1) and (4.2)

Defining

$$
\begin{aligned}
x_1 &= \operatorname{col}[\theta_{11}, \dot{\theta}_{11}, \theta_{12}, \dot{\theta}_{12}], \quad y_1 = \theta_{12} \\
x_2 &= \operatorname{col}[\theta_2, \dot{\theta}_2], \quad y_2 = \theta_2
\end{aligned}
$$

(4.1) and (4.2) can be rewritten in state form as

$$
\begin{aligned}
\mathcal{S}_1 : \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{14} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{21}^1 & -a_{22}^1 & -a_{23}^1 & -a_{24}^1 \\ 0 & 0 & 0 & 1 \\ a_{41}^1 & a_{42}^1 & -a_{43}^1 & -a_{44}^1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2^1 \\ 0 \\ 0 \end{bmatrix} u_1 \\
&+ \begin{bmatrix} 0 \\ d_2^1 \sin x_{11} \\ 0 \\ d_{41}^1 \sin x_{13} - d_{42}^1 (\tan x_{13} - \tan x_{21}) \end{bmatrix} \qquad (4.3)
\end{aligned}
$$

$$\mathcal{S}_2 : \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a^2_{21} & -a^2_{22} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ b^2_2 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ d^2_{21} \sin x_{21} + d^2_{22}(\tan x_{13} - \tan x_{21}) \end{bmatrix} \tag{4.4}$$

where

$$\begin{aligned} a^1_{21} &= \frac{k_{11}+k_{1c}}{m_{11}l^2_{11}}, \quad a^1_{22} = \frac{b_{11}+b_{1c}}{m_{11}l^2_{11}}, \quad a^1_{23} = \frac{k_{1c}}{m_{11}l^2_{11}}, \quad a^1_{24} = \frac{b_{1c}}{m_{11}l^2_{11}} \\ a^1_{41} &= \frac{k_{1c}}{m_{12}l^2_{12}}, \quad a^1_{42} = \frac{b_{1c}}{m_{12}l^2_{12}}, \quad a^1_{43} = \frac{k_{12}+k_{1c}}{m_{12}l^2_{12}}, \quad a^1_{44} = \frac{b_{12}+b_{1c}}{m_{12}l^2_{12}} \\ b^1_2 &= \frac{1}{m_{11}l^2_{11}}, \quad d^1_{41} = \frac{g}{l_{12}}, \quad d^1_{42} = \frac{k_c}{m_{12}l^2_{12}} \end{aligned}$$

and

$$\begin{aligned} a^2_{21} &= \frac{k_2}{m_2 l^2_2}, \quad a^2_{22} = \frac{b_2}{m_2 l^2_2}, \\ b^2_2 &= \frac{1}{m_2 l^2_2}, \quad d^2_{21} = \frac{g}{l_2}, \quad d^2_{22} = \frac{k_c}{m_2 l^2_2} \end{aligned} \tag{4.5}$$

Decoupled subsystems have the transfer functions

$$H_1(s) = b^1_2 \frac{a^1_{42}s + a^1_{41}}{s^4 + \dots} \tag{4.6}$$

and

$$H_2(s) = b^2_2 \frac{1}{s^2 + \dots} \tag{4.7}$$

from which we observe that

$$m_1 = \begin{cases} 4, & a^1_{42} = 0 \\ 3, & a^1_{42} \neq 0 \end{cases}$$

and

$$m_2 = 2$$

Note that, if $a^1_{42} \neq 0$, then for $H_1(s)$ to have a stable zero, we need $a^1_{41}/a^1_{42} > 0$.

For illustration purposes, let us assume

$$\begin{aligned} a^1_{41} = a^1_{42} = a^1_{43} = a^1_{44} = b^1_2 = b^2_2 = 1 \\ a^1_{21} = a^1_{22} = a^1_{23} = a^1_{24} = a^2_{21} = a^2_{22} = 0 \end{aligned}$$

Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are described by

$$\mathcal{S}_i : \left.\begin{aligned} &\dot{x}_i = A_i x_i + b_i u_i + b_i g_i(x) + h_i(y) \\ &y_i = c_i^T x_i \end{aligned}\right\} i = 1, 2$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$c_2^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$g_1(x) = a_{21}^1 x_{11} + a_{22}^1 x_{12} - a_{23}^1 x_{13} - a_{24}^1 x_{14} + d_2^1 \sin x_{11}$$

$$g_2(x) = 0$$

$$h_1(g) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{41}^1 \sin y_1 - d_{42}^1 (\tan y_1 - \tan y_2) \end{bmatrix}$$

$$h_2(y) = \begin{bmatrix} 0 \\ d_{21}^2 \sin y_2 + d_{22}^2 (\tan y_1 - \tan y_2) \end{bmatrix}$$

With this choice of parameters, (A_i, b_i, c_i^T) are controllable and observable and

$$H_1(s) = \frac{s+1}{s^2(s^2+s+1)}, \quad H_2(s) = \frac{1}{s^2}$$

Since $m_1 = 3$ and $m_2 = 2$, we have $\nu = 2$, $\nu_1 = 1$ and $\nu_2 = 2$. Therefore,

$$T_{1k} = T_k = \frac{1}{I_k} \quad \text{and} \quad T_{2k} = \tau_k = \frac{1}{I_k^2}$$

We calculate

$$e^{A_{f1}} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_{f1} = \int_0^1 e^{A_{f1}t} b_{f1} dt = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

and

$$e^{A_{f2}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_{f2} = \int_0^1 e^{A_{f2}t} b_{f2} dt = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

We choose the controller parameters as

$$A_{c1} = \begin{bmatrix} 0 & -0.1244 \\ 1 & -0.4222 \end{bmatrix}, \quad b_{c1} = \begin{bmatrix} 0.5756 \\ -2.0472 \end{bmatrix}$$

$$c_{c1}^T = \begin{bmatrix} 0 & -1.0667 \end{bmatrix}, \quad d_{c1} = -1.0667$$

to place the eigenvalues of $\hat{\Phi}_{f1}$ at

$$z_{1,2} = 0.8 \mp j0.4, \quad z_{3,4} = 0.4 \mp j0.2, \quad z_5 = 0$$

and

$$A_{c2} = -0.15, \quad b_{c2} = 0.75$$

$$c_{c2} = 0.5, \quad d_{c2} = -0.5$$

to place the eigenvalues of $\hat{\Phi}_{f2}$ at

$$z_{1,2} = 0.8 \mp j0.4, \quad z_3 = 0$$

quite arbitrarily.

At this point, we note that

$$H_{fi}(z) = c_{fi}^T (zI - e^{A_{fi}})^{-1} \Gamma_{fi} = Z\{\frac{1}{s^{m_i}}\} = \frac{q_{fi}(z)}{d_{fi}(z)}$$

as can be verified by observing that

$$c_{f1}^T (zI - e^{A_{f1}})^{-1} \Gamma_{f1} = \frac{1}{6} \frac{z^2 + 4z + 1}{(z-1)^3} = Z\{\frac{1}{s^3}\}$$

and that

$$c_{f2}^T(zI - e^{A_{f2}})^{-1}\Gamma_{f2} = \frac{1}{2}\frac{z+1}{(z-1)^2} = Z\{\frac{1}{s^2}\}$$

This observation allows us to design the local controllers in z-domain: If

$$H_{ci}(z) = c_{ci}^T(zI - A_{ci})^{-1}b_{ci} + d_{ci} = \frac{q_{ci}(z)}{d_{ci}(z)}$$

then the eigenvalues of $\hat{\Phi}_{fi}$ are the zeros of the associated closed-loop characteristic polynomial

$$\hat{d}_{fi}(z) = d_{fi}(z)d_{ci}(z) - q_{fi}(z)q_{ci}(z)$$

Once d_{ci} and q_{ci} are determined to assign the zeros of $\hat{d}_{fi}(z)$ desired values, $(A_{ci}, b_{ci}, c_{ci}^T, d_{ci})$ are found by a suitable realization of $H_{ci}(z)$. This is exactly what we did above, where we used an observable canonical realization of $H_{ci}(z)$ to obtain $(A_{c1}, b_{c1}, c_{c1}^T, d_{c1})$.

The closed-loop system is simulated with a computer program, which employs full nonlinear model of the system and uses 4-step Runga-Kutta method with a step size $h \approx 0.001$.(Actually in each common sampling interval a different step size $h_k \approx 0.001$ is used to have an integral number of h_k in τ_k. For example, when $I_k = 4$, which corresponds to a $\tau_k = 1/16$, step size is chosen to be $h_k = 1/992$ so that $\tau_k = 62h_k$.)

Arbitrary initial conditions are chosen as $x_{11}(0) = 0.2$, $x_{13}(0) = 0.1$, $x_{12}(0) = x_{14}(0) = 0$, $x_{21}(0) = 0.3$, $x_{22}(0) = 0$, and $I_k = 2$. That is, all three penduli start from rest and displaced from their vertical equilibria. The results shown in Figure 4.2-4.5 indicate that proposed adaptive, decentralized sampled-data controllers stabilize the system within a reasonable time interval of about 6 sec. From Figure 4.2, we observe that I_k is stabilized at $I_\infty = 6$, resulting in steady local sampling intervals of $T_{1\infty} = 1/6$ and $T_{2\infty} = 1/36$ and corresponding local gains $\rho_{1k} = 6$ and $\rho_{2k} = 36$. Inputs shown in Figure 4.3 indicate that controller gains are not excessively high to result in unacceptable input levels.

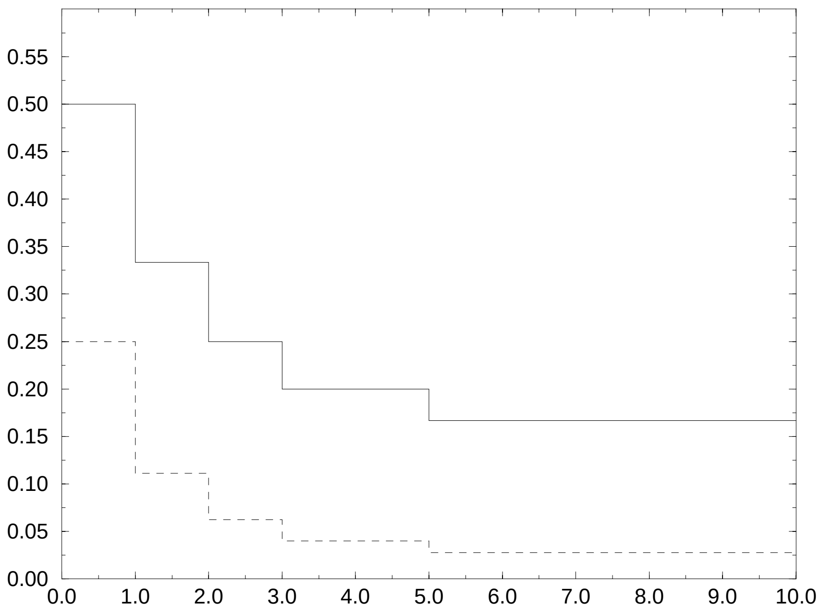

Figure 4.2: Subsystem sampling intervals: T_{1k}(solid), T_{2k}(dashed)

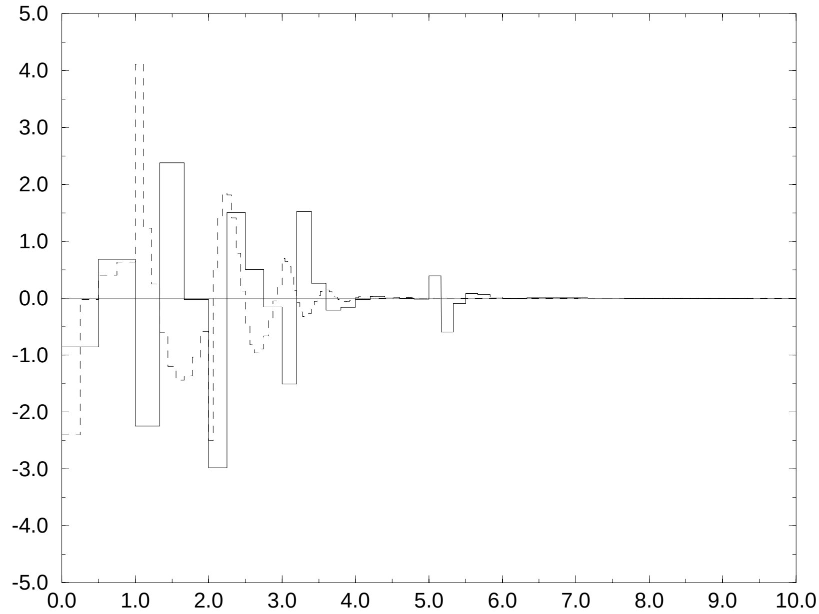

Figure 4.3: Inputs: u_1(solid), u_2(dashed)

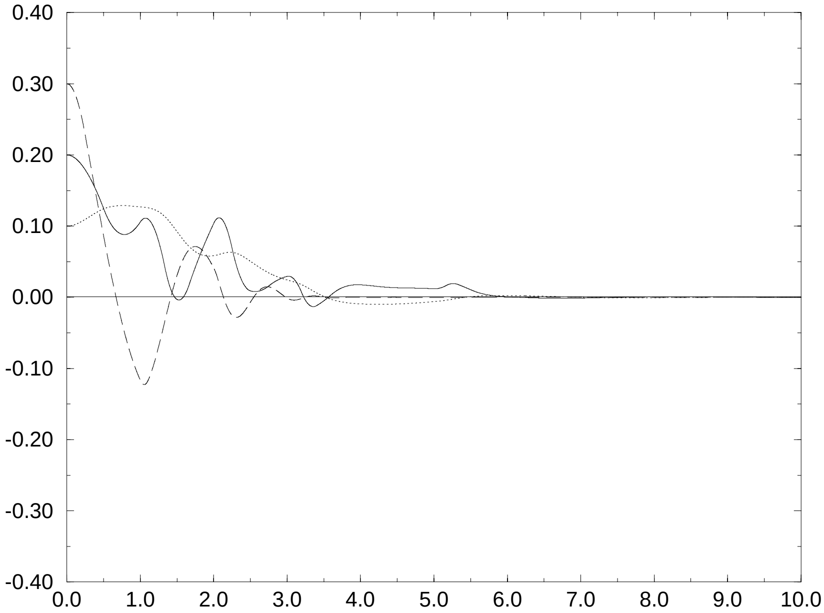

Figure 4.4: States: x_{11}(solid), x_{13}(dotted) and x_{21}(dashed)

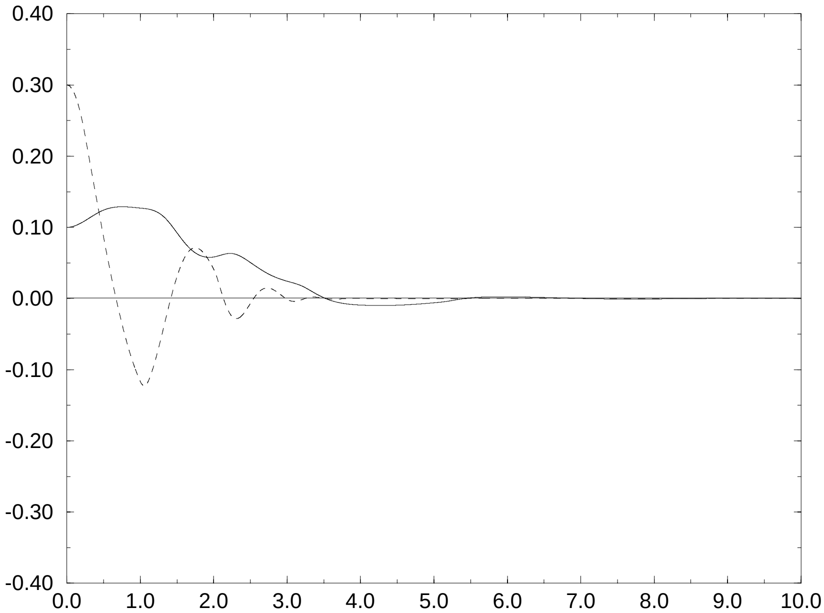

Figure 4.5: Outputs: y_1(solid), y_2(dashed)

Chapter 5

CONCLUSION

In this thesis, stabilization scheme of interconnected systems by using high-gain, decentralized and sampled-data controllers is worked on. For structured interconnections, it is shown that overall system achieves stability with fast sampling rates of controllers.

In Chapter 2, important high-gain applications are reviewed to prepare the necessary background for the main problem. The investigation is started by stating the controllable canonical forms that are the backbone of the system representation in all high-gain problems throughout the thesis. For the simplest case, single input system is stabilized by using high-gain constant state feedback controllers. Then single-input/single-output (SISO) systems are considered with high-gain dynamic output feedback controllers. In the next step, instead of continuous-time, sampled-data controllers are employed. Then, interconnected systems are examined by combining decentralized and high-gain control techniques. In each case, against unknown bounds of uncertainties, an appropriate adaptation mechanism is employed to adjust the gain accordingly.

In Chapter 3, sampled-data controllers are applied to interconnected systems, where interconnections are assumed as the major perturbations. In each subsystem, sampling interval of controller is chosen as the reciprocal of the gain. However, for overall stability, all controllers should be synchronized. Therefore, an overall gain is defined and all gains of subsystems are related to this parameter according to their relative degrees. By this way, all subsystem controllers are synchronized on a common sampling period which is an integer multiple of each subsystem period. Overall gain (naturally overall sampling period) changes with time for adaptive adjustment. In case of unknown perturbation bounds, an adaptation action is applied to decrease the sampling rate sufficiently. To protect from indefinitely decreasing sampling period, overall gain is kept unchanged for a fixed period of time.

Simulation of the proposed control methodology is presented on a spring connected inverted penduli system, in Chapter 4. By choosing arbitrary initial conditions, overall system is stabilized in a reasonable time.

High-gain has been used for stabilization of a variety of systems with uncertainties. To be able to apply this technique, the system should be combination of a controllable and observable nominal system and additive perturbations which satisfy the matching conditions. As a further research area, high-gain can be applied to perturbed systems with more general uncertainties.

In sampled-data output feedback case, we have defined gain as the reciprocal of sampling period. Although this simplifies the stability analysis, we lose degree of freedom by manipulating one parameter instead of two. Employing new relations between gain and sampling periods for other types of uncertainties can be another topic to investigate.

By decreasing the sampling intervals sufficiently, we have obtained overall stability of the system. If we keep these sampling rates, the stability will be preserved. Decreasing sampling rates without disturbing the stability seems possible for some systems. As a further work, these systems and increment margin of sampling rates can be explored.

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